

Discrete Mathematics - Übungsblatt & Tutorium

Martina JUHNKE, Tarek EMMERICH, Germain POULLOT

April - July 2025

Jedes Übungsblatt enthält die Übungen (in der Regel Übungen 1 bis 4 und Bonusübung)

Ich (Germain) schreibe die Übungen auf Englisch und übersetze sie dann ins Deutsche (mit ChatGPT). Wenn du also das Gefühl hast, dass etwas seltsam ist oder fehlt, schau dir bitte die englische Version an.

Nicht alle Übungen sind korrigiert: für die verbleibenden sollten Sie zu den Mittwochssitzungen kommen!

Each exercise sheet contains the exercises of the Übungsblatt (usually exercises 1 to 4 and bonus exercise)

I (Germain) am writing the exercises in English and then doing the translation to German (with ChatGPT), so if you feel like something is weird or lacking, go to the English version, please.

Not all exercises are corrected: for the remaining one, you should come the the Wednesday' sessions!

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Discrete Mathematics – Summer Semester 2025

Exercise Sheet 0

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1

[General Information]

Submit a sheet confirming that you have understood the following information.

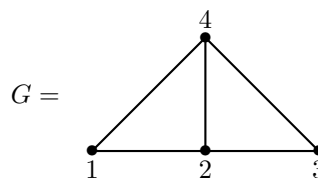
1. Lectures are on Tuesdays and Thursdays, 10 am to 12 pm, room 66/E33.
2. The exercise session is held on Thursdays from 12 pm to 2 pm in room 69/117. Solutions to the exercise sheets will be presented during the exercise session. The first exercise session will take place on 24.04.24.
3. Exercise sheets will be published weekly on Tuesdays. To be eligible to take the exam, you must achieve 50% of the points on the exercise sheets and quick series of questions (50% on each of them, not in total!).
4. Exercise sheets must be submitted online via Vips. Submission is possible from 10 am on the Tuesday one week before the deadline (the next Wednesday at 11:59 am). Exercise sheets can be submitted in groups of up to three students. To do this, sign up for one of the groups under Vips → Groups. Submission counts when one person from the group submits the task (in case of doubt, the last uploaded file counts). Preferably upload solutions as pdf written with LaTeX, or otherwise as jpg (for example, if hand-written). Corrections will then be visible under Vips → Results.
5. From one Thursday to the next, you must answer, individually, a quick series of questions (time limit: 90 minutes), on Vips.
6. Additionally, two tutorials are offered: Thursdays from 2 pm to 4 pm (in 15/E07), and Fridays from 12 am to 2 pm (in 93/E31). The tutorials offer additional exercise problems to help you work on the exercise sheets. You can also ask questions about the exercise sheets in the tutorials. The tutorials are an optional additional offer. If you prefer to speak English/français/español, please come on Thursdays.

Exercise 2

[Adjacency Matrix and paths]

Compute the adjacency matrix A of the graph G drawn here.

Compute A^2 and A^3 . Show that the entry in line i and column j of A^2 (respectively A^3) is the number of paths of length 2 (resp. 3) between i and j .

**Exercise 3**

[Graph on graphs]

Draw the graph \mathcal{G}_3 with:

- Vertices: all the graphs on (exactly) 3 vertices,
- Edges: there is an edge between a graph G and a graph H if G is a sub-graph of H (or the opposite).

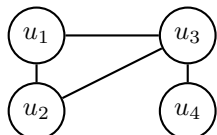
How many vertices does \mathcal{G}_3 have? How many edges does \mathcal{G}_3 have? Is there a sub-graph of \mathcal{G}_3 which is a cycle containing all its vertices?

Exercise 4

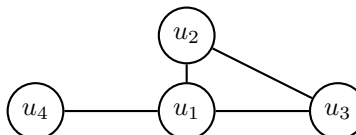
[Graph Isomorphisms]

Decide whether the following graphs are isomorphic.

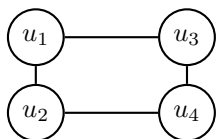
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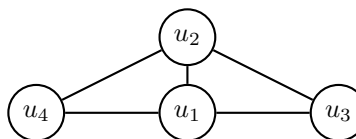
and



2.



und



If yes, provide a graph isomorphism.

If not, explain why not.

Exercise 5

[Bonus exercise – A Wikipedia game]

Consider the graph whose vertices are all (English) Wikipedia pages, and whose edges are between two pages if there is a link from one page to the other (i.e. you can click on a blue word on one page to go to the other). In this graph, what is the distance between the page “Graph (discrete mathematics)” and the page “Universe”?

Submission: 23.04.2025 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

ChatGPT told me this (it is a real copy-paste from the free version of ChatGPT):
“ Let $p_1, \dots, p_n \in \mathbb{R}^2$ be n points, and define a graph G where $\{p_i, p_j\}$ is an edge if either p_j is the closest point to p_i , or vice versa.

The minimum number of (undirected) edges in G is $\boxed{n-1}$.

Example: if all p_i ($i \geq 2$) are closest to a central point p_1 , and p_1 has one closest neighbor among them, then each pair contributes at most one edge, yielding $n-1$ edges in total. ”

This is false, explain me why.

Exercise 7

[Drawing some graphs]

If possible, draw each of the following graphs, estimate the number of vertices, edges, connected components..., and decide if it is a tree or not. Some graphs are not properly defined, it is voluntary: you need to find which graph the author would have like to define.

1. vertices: all countries in the European Union; edges: between two countries sharing a border (use a map from Internet).
2. vertices: all rooms in the university; edges: between rooms that share a door.
3. vertices: n points $\mathbf{p}_1, \dots, \mathbf{p}_n$ in the plane (none of them being equidistant to two others); edges: $(\mathbf{p}_i, \mathbf{p}_j)$ if j is the index that minimizes $\|\mathbf{p}_i - \mathbf{p}_k\|$ for $k \neq i$.
4. vertices: numbers from 0 to 16; edges: (x, y) if x divides y , and there is no z such that x divides z and z divides y .
5. vertices: all graphs on 4 vertices; edges: (G, H) if H is a sub-graph of G .
6. vertices: pairs (letter of the alphabet, number); edges: between (X, i) and (Y, j) if $j = i + 1$ and there exists a word (in the German dictionary) with the letter X in i -th position, and the letter Y in j -th position (draw it for a small dictionary).
7. vertices: all proven mathematical theorems (or all math courses you have followed during your life); edges: from a theorem to another if the second one uses the first one to be proven.
8. vertices: all maths tutorial sessions that are given this semester; edges: between session whose scheduled time slots intersect (*show that all cycles of length $k \geq 4$ have a chord, i.e. two vertices, non-adjacent in the cycle, that share an edge*).
9. vertices: all Wikipedia pages; edges: between two pages if there is a link from one page to the other.

10. vertices: all the pieces of an IKEA furniture; edges: between two pieces if you need to fix one to the other when assembling the furniture.

Exercise 8 [Too many edges forces the graph to be connected]
Let G be a graph on n vertices, with (strictly) more than $\binom{n-1}{2} = \frac{(n-2)(n-1)}{2}$ edges. Show that G is connected.

Exercise 9 [Complementary graph and connectivity]
For a graph $G = (V, E)$, let its complementary graph be $\overline{G} = (V, \overline{E})$, where there is an edge uv in \overline{G} if and only if there is no edge uv in G . If G is not connected, show that \overline{G} is connected.

Discrete Mathematics – Solutions Sheet 0

Exercise 1

[General information]

Well done, you know how to read!

Exercise 2

[Adjacency matrix and paths]

The adjacency matrix of the graph G is $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$, in line i and column j , we put a 1 if there is an edge ij in the graph, otherwise we put a 0. We have:

$$A^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix} \quad \text{and} \quad A^3 = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}$$

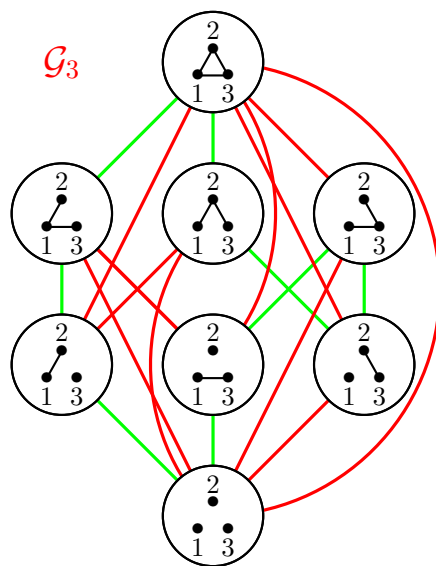
The reader easily sees that there are 5 paths of length 3 (i.e. with 3 edges and 4 vertices) from the vertex 1 to the vertex 2 in G : $(1, 4, 3, 2)$, $(1, 2, 3, 2)$, $(1, 2, 4, 2)$, $(1, 4, 1, 2)$, $(1, 2, 1, 2)$. The same holds for all the other entries of A^2 and A^3 .

However, note that we are not really counting “paths”, but rather “walks”: a path can only use some edge once, whereas a walk can use several times the same edge. The problem with taking powers of an adjacency matrix is that it does not grant us access to the number of paths, but only of walks (which are less interesting usually).

Exercise 3

[Graph on graphs]

In \mathcal{G}_3 , there are 8 vertices and 19 edges. There is a cycle inside \mathcal{G}_3 which contains all its vertices, we draw it in green on the figure.



Exercise 4

[Graph Isomorphism]

1. These graphs are isomorphic, via $u_1 \mapsto u_3, u_2 \mapsto u_2, u_3 \mapsto u_1, u_4 \mapsto u_4$
2. These two graphs are not isomorphic: they do not have the same number of edges.

Exercise 5

[Bonus – A Wikipedia game]

The length is 4 (maybe less):

Graph (discrete mathematics) \rightarrow Empty set \rightarrow Nothing \rightarrow Space \rightarrow Universe
 If you have find any other path that is not “too long” (maybe even shorter), be happy!

Exercise 6

[With ChatGPT]

The minimum is $\lceil \frac{n}{2} \rceil$, not $n - 1$. Indeed (will do only the case where n is even), take the points $(0, 0), (1, 0), (2, 0), (2, 1), (4, 0), (4, 1), \dots, (2n, 0), (2n, 1)$. Then the edges of the resulting graph are between $(2i, 0)$ and $(2i, 1)$. There n edges, but $2n$ vertices, so the number of edges is half the number of vertices.

Exercise 7

[Drawing some graphs]

Exercise 8

[Too many edges forces the graph to be connected]

Recall that the maximum number of edges of a graph on n vertices is $\binom{n}{2}$ (which is the number of edges of the complete graph).

Suppose G is not connected. Then there exists $A, B \subsetneq V$ with $A \cap B = \emptyset$ and $A \cup B = V$, such that there is no edge in G between (a vertex in) A and (a vertex in) B . We denote $|A| = a$ and $|B| = b$, with $a + b = n$ and $1 \leq a, b \leq n - 1$.

In particular, the number of edges of G is at most $\binom{a}{2} + \binom{b}{2} = \frac{1}{2}(a(a-1) + b(b-1)) = \frac{1}{2}(a^2 + b^2 - (a+b))$. This quantity is maximized when $a = n-1$ and $b = 1$ (or the converse), giving a maximum of $\binom{n-1}{2}$.

Hence, we have proven that if G has strictly more than $\binom{n-1}{2}$, then it is connected.

Exercise 9

[Complementary graph and connectivity]

Let $G = (V, E)$ be a disconnected graph. Then there exists $A, B \subsetneq V$ with $A \cap B = \emptyset$ and $A \cup B = V$, such that there is no edge in G between (a vertex in) A and (a vertex in) B .

Consequently, for $x \in A$ and $y \in B$, there is an edge in \overline{G} . Thus, \overline{G} contains a connected sub-graph on all its vertices, namely $(A \cup B, \{(x, y) ; x \in A, y \in B\})$ (this graph is called the complete bipartite graph on A, B).

Discrete Mathematics – Summer Semester 2025

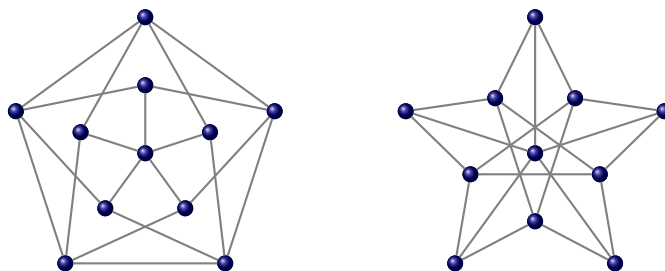
Exercise Sheet 1

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

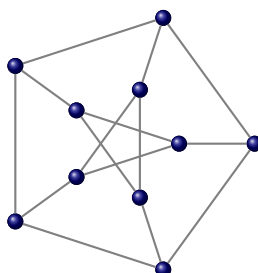
Exercise 1

[Isomorphism of Graphs]

- (i) Determine whether the following two graphs are isomorphic. Justify your answer and provide an isomorphism if applicable.



- (ii) Show that the graph in the lower illustration is isomorphic to the graph whose vertices are all the pairs of numbers $\{i, j\}$ with $1 \leq i \leq 5$, and $1 \leq j \leq 5$; and where two such nodes $\{i, j\}, \{k, l\}$ form an edge if and only if $\{i, j\} \cap \{k, l\} = \emptyset$.



Exercise 2

[Trees]

Show that the following statements for a non-empty graph $G = (V, E)$ are equivalent:

- (i) G is a tree.
- (ii) G is *minimally connected*, i.e., G is connected and for each edge $e \in E$ the graph $G \setminus e := (V(G), E(G) \setminus \{e\})$ is not connected.
- (iii) G is *maximally cycle-free*, i.e., G contains no cycles and for any two non-adjacent vertices u and v in V , the graph $G + uv := (V, E \cup \{uv\})$ contains a cycle.

Exercise 3

[Center of trees]

A vertex $u \in V$ is a *center* of a graph $G = (V, E)$ when $\max\{\text{dist}(u, v) ; v \in V\}$ is as small as possible.

- (i) Let T be a tree with at least 3 vertices, and T' the tree obtained by deleting every leaf of T . Show that T and T' have the same centers.
- (ii) Deduce that a tree has either a unique center or two adjacent centers.
- (iii) Give a graph on n vertices with n centers.

Exercise 4

[Spanning Trees]

Let $G = (V, E)$ be a graph, T a spanning tree in G , and C a cycle in G .

- (i) Let $e \in E(T) \cap E(C)$ be an edge. Show that there exists an edge $f \in E(C) \setminus \{e\}$ such that $(T \setminus e) + f$ is a spanning tree.
- (ii) Let T' be another spanning tree in G , and let $e \in E(T) \setminus E(T')$ be an edge. Show that there exists an edge $f \in E(T')$ such that $(T \setminus e) + f$ is a spanning tree.

Exercise 5

[Bonus exercise – Chemistry]

Show that the graphical representation of alkanes (hydrocarbons) of formula C_nH_{2n+2} is a tree (remember that the carbon atom is valence 4, and the hydrogen atom is valence 1).

For n from 3 to 6, how many different isomers of formula C_nH_{2n+2} are there? Draw them. Recall that two molecules are isomers when they have the same formula but not the same graphical representation (carbon atoms are indistinguishable from each other, and so are hydrogen atoms).

Submission: by 30.04.2025 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

ChatGPT told me this (it is a real copy-paste from the free version of ChatGPT):

“ To maximize the number of cycles in a simple graph with n vertices and $n + 1$ edges, consider the following construction:

Take the complete graph K_{n-1} , which has $\binom{n-1}{2}$ edges and contains a large number of cycles. Then, add a new vertex and connect it to exactly two of the existing vertices in K_{n-1} .

This gives a graph with:

- n vertices,
- $n + 1 = \binom{n-1}{2} + 2$ edges,
- and at least $\binom{n-1}{2}$ distinct cycles, all of which are already present in K_{n-1} .

Hence, the maximum number of cycles in a simple graph with n vertices and $n + 1$ edges is: $\boxed{\binom{n-1}{2}}$. ”

That’s false, explain why. What is the maximum number of cycles of a graph with n vertices and $n + 1$ edges?

Exercise 7

[Drawing some graphs]

(Copy-pasted from Tutorium 0, Exercise 7)

If possible, draw each of the following graphs, estimate the number of vertices, edges, connected components..., and decide if it is a tree or not. Some graphs are not properly defined, it is voluntary: you need to find which graph the author would have like to define.

1. vertices: all countries in the European Union; edges: between two countries sharing a border (use a map from Internet).
2. vertices: all rooms in the university; edges: between rooms that share a door.
3. vertices: n points $\mathbf{p}_1, \dots, \mathbf{p}_n$ in the plane (none of them being equidistant to two others); edges: $(\mathbf{p}_i, \mathbf{p}_j)$ if j is the index that minimizes $\|\mathbf{p}_i - \mathbf{p}_k\|$ for $k \neq i$.
4. vertices: numbers from 0 to 16; edges: (x, y) if x divides y , and there is no z such that x divides z and z divides y .
5. vertices: all graphs on 4 vertices; edges: (G, H) if H is a sub-graph of G .
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7. vertices: all proven mathematical theorems (or all math courses you have followed during your life); edges: from a theorem to another if the second one uses the first one to be proven.
8. vertices: all maths tutorial sessions that are given this semester; edges: between session whose scheduled time slots intersect (*show that all cycles of length $k \geq 4$ have a chord, i.e. two vertices, non-adjacent in the cycle, that share an edge*).
9. vertices: all Wikipedia pages; edges: between two pages if there is a link from one page to the other.
10. vertices: all the pieces of an IKEA furniture; edges: between two pieces if you need to fix one to the other when assembling the furniture.

Exercise 8

[Rehearsal on trees]

- (i) Show that contracting an edge of a tree yields a tree. Show that deleting a leaf of tree yields a tree.
- (ii) Show that a tree is 2-colorable (i.e. can be colored by black and white such that no two adjacent vertices receive the same color).
- (iii) Show that between any two vertices of a tree, there exists a unique path.

Exercise 9

[Unicity of minimal spanning tree]

Let $G = (V, E)$ be a connected graph, and $\omega : E \rightarrow \mathbb{R}_+$ a weight function. We want to prove that: *if $\omega(e) \neq \omega(f)$ for all edges $e \neq f$, then G has minimal spanning tree; moreover, this tree contains the edge with the smallest weight.*

- (i) Re-read the last exercise on spanning trees.
- (ii) Conclude that a minimal spanning tree always contain the minimal edge.
- (iii) Suppose there are two minimal spanning trees T_1 and T_2 of G . Consider the edge e with minimal weight that is in T_1 but not in T_2 or in T_2 but not in T_1 . Suppose e is in T_1 . Construct a spanning tree of G with a smaller weight.
- (iv) Conclude on the unicity of the minimal spanning tree.

Exercise 10

[Prüfer sequences]

Take a tree T on vertices labelled from 1 to n . The *Prüfer sequence* p_T is a sequence of n numbers in $\{1, \dots, n\}$ associated to T is obtained inductively. Let i be the leaf of T with the minimal label, $T \setminus i$ the tree obtained from T by removing i , and j_i the unique neighbor of i in T , then: $p_T = j \cup p_{T \setminus i}$.

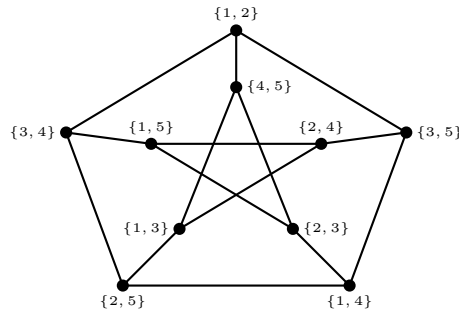
- (i) Write the Prüfer sequences for all labeled trees up to 4 vertices.
- (ii) Show that $T \mapsto p_T$ is injective. Show that $T \mapsto p_T$ is surjective (on which set?). (Hint: Construct a reciprocal of the map $T \mapsto p_T$.)
- (iii) How many labeled trees on n vertices are there ?

Discrete Mathematics – Solutions Sheet 1

Exercise 1

□

- (i) Yes, both graphs are isomorphic.
- (ii) We provide below a labeling of the Petersen graph, from which it is directly visible that it is isomorphic to the graph with vertex set $\binom{\{1, \dots, 5\}}{2}$, where two such vertices $\{i, j\}, \{k, l\}$ form an edge if and only if $\{i, j\} \cap \{k, l\} = \emptyset$.



Exercise 2

[Trees]

(i) \Rightarrow (ii): Since G is a tree, its connectivity directly follows. Now, we show that $G \setminus e$ is not connected for every edge $e \in E(G)$. Suppose there exists $e = vw \in E(G)$ such that $G - e$ is connected. Then there exists a v - w path P in $G \setminus e$. Adding the edge e to P would then form a cycle in G , contradicting the fact that G is a tree and hence acyclic.

(ii) \Rightarrow (i): Since G is assumed to be connected, we only need to show that G does not contain cycles. If G had a cycle C , then $G \setminus e$ would be connected for every edge $e \in E(C)$, contradicting the assumption.

(i) \Rightarrow (iii): Since G is a tree, we already know that G is acyclic. Let $u, v \in V(G)$ such that $uv \notin E(G)$. Since G is connected, there exists a u - v path $P = uu_1 \dots u_{s-1}v$ in G . Then $uu_1 \dots u_{s-1}vu$ forms a cycle in $G \cup uv$.

(iii) \Rightarrow (i): Since G is assumed to be cycle-free, it suffices to show that G is connected. Let $u, v \in V(G)$. If $uv \in E(G)$, then uv is already a u - v path. If $uv \notin E(G)$, then $G' = (V, E \cup uv)$ contains a cycle

$$uu_1 \dots u_{s-1}vu.$$

In this case, $uu_1 \dots u_{s-1}v$ is a u - v path in G .

Exercise 3

[Center of trees]

For $u \in V$, we denote by $\delta_T(u) = \max_{v \in V} \text{dist}(u, v)$. Note that $\delta_T(u) \geq 1$ because there are at least 2 different vertices in T .

If u is a leaf of T , we denote v_u^T its unique neighbor in T . For a leaf u of T , then $\text{dist}(v_u^T, w) = \text{dist}(u, w) - 1$ for $w \neq u$ (and $\text{dist}(v_u^T, u) = 1$), so $\delta_T(v_u^T) = \delta_T(u) - 1$.

Furthermore, $\delta_T(u)$ is achieved on a leaf of T : it is true for the tree on 3 vertices, and if it is true on T' , then it is true on T because for all $w \in T$ and u a leaf of T , we have $\text{dist}(w, u) = \text{dist}(w, v_u^T) + 1$, where v_u^T is a leaf of T' .

- (i) First, if u is a center of T , then it is not a leaf of T , because if u is a leaf, then $\delta_T(v_u^T) < \delta_T(u)$. Besides, consider any vertex w of T that is not a leaf: $\delta_T(w) = \text{dist}(w, u)$ for some leaf u of T , thus $\text{dist}(w, v_u^T) = \delta_T(w) - 1$ is the maximum of $\text{dist}(w, v)$ for $v \in T'$, i.e. $\delta_{T'}(w) = \delta_T(w) - 1$. Consequently, the minimizers of δ_T and $\delta_{T'}$ are the same: T and T' have the same centers.
- (ii) We do an induction on the number of vertices of T . If T has 1 vertex, it has a unique center. If T has 2 vertices, then both are centers and they are adjacent. If T has 3 or more vertices, then T has the same centers than T' (obtained from T by removing all its leaves): as T' has strictly fewer vertices, T' has 1 center or 2 adjacent centers, and the induction is complete.

Exercise 4

[Spanning Trees]

- $T \setminus e$ is not connected. Let $e = vw$. Since C is a cycle, there exists a path P from v to w in $C \setminus e$. Let A and B be the connected components of $T \setminus e$. Since T is a spanning tree, there exists an edge $f = xy \in E(P)$, with $x \in V(A)$ and $y \in V(B)$. Adding f to $T \setminus e$ creates a spanning tree, because $(T \setminus e) + f$ is minimally connected and spanning.
- Let $e = vw$. Since T' is a spanning tree, there exists a path $P \subseteq T'$ from v to w . $P + e$ forms a cycle. The claim follows from (i).

Exercise 5

[Bonus – Chemistry]

Consider the graph $G = (V, E)$ whose vertices are the atoms of the molecule C_nH_{2n+2} , and whose edges are the bonds between the atoms. This graph has $n + (2n + 2) = 3n + 2$ vertices, with n vertices of degree 4, and $2n + 2$ vertices of degree 1. The degree formula grants that $2 \#E = 4 \times n + 1 \times (2n + 2) = 6n + 2$. Hence $\#E = 3n + 1 = \#V - 1$. As this is the graph of a molecule, it is connected, so it is a tree. Note that it is impossible that there is a double-bond between carbon atoms because it would break the connectivity of the molecule.

As hydrogen atoms are indistinguishable, and carbon atoms are indistinguishable, we can just represent a molecule by its carbon atoms. We can construct all these trees by adding the carbon atoms one at a time:

- For $n = 3$, there is 1 tree.
- For $n = 4$, there are 2 trees.

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[With ChatGPT]

Exercise 7

[Drawing some graphs]

[Rehearsal on trees]

(i) Contracting yields a connected graph and the number of edges is still 1 less than the number of vertices. Deleting a leaf diminish the number of vertices by 1, and the number of edges also by 1, and the connectivity is kept.

- (ii) Suppose all trees on n vertices are 2-colorable (this is true for $n = 1$ and $n = 2$). Take a tree T on $n+1$ vertices, remove a leaf v , 2-color the resulting tree. In T , the leaf v has only 1 neighbor u : color v in the opposite color of u to get a 2-coloring of T . By induction, all trees are 2-colorable.
- (iii) Idem : this is true for trees with $n = 1$ and $n = 2$ vertices. If this is true for all trees on n vertices, then take T on $n+1$ vertices, discard a leaf v to obtain T' . Let u, u' be vertices of T . If u, u' are vertices of T' , then there is a unique path between them (by induction), otherwise $u = v$, and let \bar{v} be the unique neighbor of u in T . All paths between v and u' gives a path between \bar{v} and u' , and we know there is a unique such path. Hence there is a unique path between v and u' .

Exercise 9

[Unicity of minimal spanning tree]

- (i)
- (ii) Suppose the minimal spanning tree T does not contain the minimal edge e . Then $T + e$ contains a cycle C , and hence there exists another spanning tree $T' = (T \setminus f) + e$ for some $f \neq e$ an edge of T . As the weight of T' is $\omega(T) + \omega(e) - \omega(f) < \omega(T)$ (because $\omega(e)$ is minimal), we have a contradiction.
- (iii) Consider $T_2 + e$: it has a cycle C , so we can remove an edge $e_2 \neq e$ from T_2 such that $T' = (T_2 \setminus e_2) + e$ is again a tree. It is spanning G , and $\omega(T') = \omega(T_2) + \omega(e) - \omega(e_2) < \omega(T_2)$ by hypothesis on e .
- (iv) If there are two different minimal spanning trees T_1 and T_2 , then the previous question allows to construct T' with $\omega(T') < \omega(T_2)$, which contradicts the minimality of T_2 . Consequently, there can not be two different minimal spanning trees.

Exercise 10

[Prüfer sequences]

- (i) There are 16 labeled trees on 4 vertices, and Prüfer sequences are all sequences of length 3 where the first 2 values can be any number in $1, \dots, 4$ (and the last digits is always 4).
- (ii) blah blah, see Wikipedia
- (iii) There are n^{n-2} sequences that can be Prüfer sequences, hence there are n^{n-2} labeled trees on n vertices.

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 2

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1

[3-regular graphs]

A graph $G = (V, E)$ is 3-regular when every vertex has exactly 3 neighbors.

1. Prove that $|V|$ is even.
2. For $p \geq 2$, construct explicitly a 3-regular graph $G = (V, E)$ with $|V| = 2p$.
(Hint: For $p \geq 3$, think bipartite.)

Exercise 2

[Grids, bicycle-wheels, complete k -partite graph]

The *waist-width* of a graph G is the length of a shortest circle in G .

The *diameter* of a graph G is the maximum length of a shortest path in G .

Determine the number of edges, maximum degree, average degree, waist width and diameter of the following graphs:

- (i) [Grid] $G_{n,m} = (V, E)$ with $V = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E = \{\{(v, w), (v', w')\} : |v - v'| + |w - w'| = 1\}$.
- (ii) [Bicycle-wheel] $BW_k = (V, E)$ with $V = \{a, b, c_1, \dots, c_k\}$ and $E = \{ab\} \cup \{ac_i, bc_i : i \in [k]\} \cup \{c_1c_2, \dots, c_{k-1}c_k, c_kc_1\}$. See Figure 1
- (iii) [Complete k -partite graph] $K_{n_1, \dots, n_k} = (V, E)$ given by $V = V_1 \cup \dots \cup V_k$ with $|V_i| = n_i$ and $E = \{vw : v \in V_i, w \in V_j \text{ with } i \neq j\}$. $A \cup B$ is the *disjoint union*, that is $A \cup B$ with $A \cap B = \emptyset$.

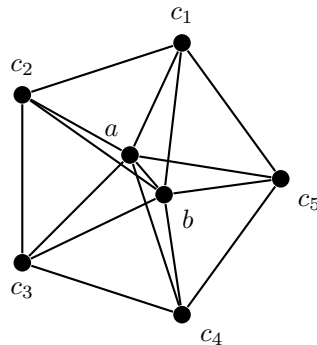


Abbildung 1: BW_5

Exercise 3

[Adjacency Matrix]

The *trace* of an $n \times n$ matrix $A = (a_{ij})$ is defined as the sum of the diagonal elements, i.e., $\text{trace}(A) = \sum_{i=1}^n a_{ii}$. Let $G = (V, E)$ be a graph with adjacency matrix A .

- (i) Show that $\text{trace}(A^2) = 2|E|$.
- (ii) Show that $\text{trace}(A^3) = 6 \times (\text{number of triangles in } G)$. Where three vertices i, j, k forms a triangles if ij, jk and ik are edges of G .
- (iii) Let B be the adjacency matrix of K_n . Find a formula for $\text{trace}(B^3)$.
- (iv) Show that $\text{trace}(B^4) = n^4 + 2n^3 + n^2$. Recall that $n^k = n(n-1)(n-2) \dots (n-k+1)$.

Exercise 4

[Bridges]

Let $G = (V, E)$ be a graph. An edge $e \in E$ is called a *bridge* of G if the number of components of $G \setminus e$ is greater than that of G .

- (i) Which trees have bridges?
- (ii) For which m, n does $K_{m,n}$ have a bridge? Justify your answer.
- (iii) Show that an edge e of a graph G is a bridge if and only if e does not belong to any cycle of G .

Exercise 5

[Bonus – Graph of friendship]

Have you noticed that, at a party, there are always (at least) two persons that know exactly the same number of other people?

Prove this fact by introducing the right graph. (Assume that if person A knows person B , then B also knows A .)

Submission: by 07.05.2025 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

ChatGPT told me this (it is a real copy-paste from the free version of ChatGPT):

“ Let $G = (V, E)$ be a simple graph with adjacency matrix A . We want to compute $\text{trace}(A^4)$.

Closed walks and powers of the adjacency matrix Recall that the entry $(A^k)_{ij}$ counts the number of walks of length k from vertex i to vertex j in G . In particular, the trace of A^k counts the number of closed walks of length k in G :

$$\text{trace}(A^k) = \sum_{i \in V} (A^k)_{ii}$$

This is the total number of closed walks of length k starting and ending at the same vertex.

Computation of $\text{trace}(A^4)$ We want to compute $\text{trace}(A^4)$, which corresponds to the total number of closed walks of length 4 in G .

There are several types of walks contributing to this trace:

1. **Edge bounces:** For each edge $\{i, j\} \in E$, there are closed walks like $i \rightarrow j \rightarrow i \rightarrow j \rightarrow i$, and its reverse. Each such pair contributes 2 closed walks of length 4. Since each edge allows such walks starting at both ends, the contribution is:

$$\text{from edges: } 2|E|$$

2. **Triangles:** Every triangle contributes additional closed walks of length 4, for example by traversing three vertices and returning with a backtrack. These contribute in total 4 closed walks per triangle.
3. **4-cycles:** Every 4-cycle (square) contributes 8 closed walks of length 4, corresponding to 4 starting points and 2 directions (clockwise and counterclockwise).

There is a well-known identity in graph theory that summarizes this:

$$\text{trace}(A^4) = 2 \sum_{i \in V} \deg(i)^2 + 4 \cdot t + 8 \cdot q$$

where:

1. $\deg(i)$ is the degree of vertex i ,
2. t is the number of triangles in the graph,
3. q is the number of 4-cycles (i.e., quadrilaterals).

„

Exercise 7

[König's lemma on bipartite graphs]

We want to prove the following [König, 1916]: *A graph is bipartite if and only if it has no cycle of odd length.*

1. Prove that a graph is bipartite if and only if it is 2-colorable (i.e. we can color the vertices with black and white such that no edge have its two endpoints of the same color).
2. Prove that if a graph is 2-colorable, then it has no odd cycle.
3. Let G be a graph without odd cycle, we want to color it with 2 colors:
 - (a) Explain why we can restrict to connected graphs.
 - (b) Suppose G is connected and fix $x \in V$. Let V_k the subset of V formed by the vertices that are at distance exactly k from x . Prove that if there is an edge between $y \in V_{2p}$ and $z \in V_{2q}$ for some p, q , then G has an odd cycle.
 - (c) Conclude

Exercise 8

[Complete graph, Path, Cycle, Circulants]

Same as Exercise 2, on the following graphs:

1. Complete graph $K_n = (V, E)$ with $V = \{1, \dots, n\}$ and $E = \{ij ; i \in V, j \in V\}$.
2. Path $P_n = (V, E)$ with $V = \{1, \dots, n\}$ and $E = \{ij ; j = i + 1\}$.
3. Cycle $C_n = (V, E)$ with $V = \{1, \dots, n\}$ and $E = \{ij ; j = i + 1 \text{ or } (i = 1, j = n)\}$.
4. Let $a_1, \dots, a_\ell \in \mathbb{N}$ with $0 < a_1 < \dots < a_\ell \leq \frac{n}{2}$. We consider the graph $C_n(a_1, \dots, a_\ell) = (V, E)$ with $V = \{v_0, v_1, \dots, v_{n-1}\}$ and $E = \{v_i v_{i+a_k} : 1 \leq k \leq \ell, 0 \leq i \leq n-1\}$ (indices are considered modulo n). See Figure 2. Determine the number of edges in $C_n(a_1, \dots, a_\ell)$.
5. Your favorite graph from Tutorium 1.

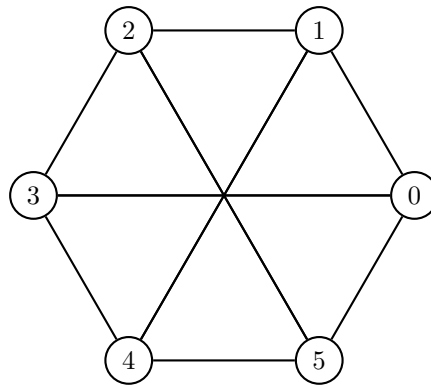


Abbildung 2: $C_6(1, 3)$

Exercise 9

[Mantel's theorem]

A *triangle* in a graph $G = (V, E)$ is a triplet of vertices $u, v, w \in V$ that are pairwise linked by an edge, i.e. $(u, v), (v, w), (u, w) \in E$. We want to prove Mantel's theorem: *If a graph on n vertices has no triangle, then it has at most $\frac{n^2}{4}$ edges.*

1. Fix n and consider $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, the complete bipartite graph. How many edges has it? Conclude that of Mantel's theorem is optimal.

2. Prove that if Mantel's inequality holds for all connected components of G then it holds for G .
3. Fix $G = (V, E)$ without triangle, with $|V| = n$. Let $d(u)$ be the *degree* of $u \in V$. Show that for an edge $(u, v) \in E$, then $d(u) + d(v) \leq n$.
4. Prove Mantel's theorem by induction on its number of vertices. *Hint:* Express the number of edges of G by fixing $(u, v) \in E$ and considering H , the graph obtained by removing the vertices u and v from G .

Exercise 10

[Connectivity when discarding a vertex]

Let G be a connected graph. We want to prove that there exists a vertex v of G such that $G \setminus \{v\}$ is connected.

1. Find a connected graph in which any vertices can be discard without destroying connectivity.
2. Find a connected graph in which only 2 different vertices can be discard without destroying connectivity.
3. Fix a connected graph G and consider the longest path (s_1, s_2, \dots, s_r) in G . Let $H = G \setminus \{s_1\}$ and fix t, t' vertices of H .
 - (a) Why is there a path (t_1, \dots, t_m) in G with $t_1 = t$ and $t_m = t'$?
 - (b) If $s_1 \neq t_j$ for all j , show that t and t' are connected in H .
 - (c) If $s_1 = t_j$ for some j , show that $t_{j-1} = s_u$ for some $u \geq 2$ and $t_{j+1} = s_v$ for some $v \geq 2$.
 - (d) Show that there exists a path from t to t' in H , and conclude.
 - (e) Show that $G \setminus \{s_r\}$ is also connected, and conclude on the minimality of the example you have found in question 2.

Discrete Mathematics – Solutions Sheet 2

Exercise 1

[3-regular graphs]

1. Degree formula : $\sum_{u \in V} d(u) = 2|E|$. For all $u \in V$, $d(u) = 3$ because G is 3-regular, hence: $3|V| = 2|E|$, so $|V|$ is even (because 3 is not).
2. One example of construction (there are a lot): $V = \{-p, -(p-1) \dots, -1\} \cup \{1, 2, \dots, p\}$, and the vertex $-i$ shares an edge with the three vertices i , $i+1$ and $i+2 \pmod{p}$. Then the vertex j shares an edge with the vertices $-j$, $-(j-1)$ and $-(j-2) \pmod{p}$, so the graph is 3-regular.

You can re-do this exercise for any q -regular graph for q prime.

Exercise 2

[Grids, bicycle-wheels, complete k -partite graph]

(i)

- $|E(BW_k)| = 3k + 1$
- $\Delta(BW_k) = k + 1$
- $\frac{\sum_{v \in V(BW_k)} d(v)}{|V(BW_k)|} = \frac{6k + 2}{k + 2}$
- The girth of BW_k is 3 for $k \geq 4$ and 2 for $k = 3$.
- The diameter of BW_k is 2 for $k \geq 4$ and 1 for $k \leq 3$.

(ii)

- $|E(G_{n,m})| = 2nm - m - n$
- The average degree of $G_{n,m}$ is as follows:

n	m	Average degree of $G_{n,m}$
1	1	0
1	≥ 2	$\frac{2m-2}{m}$
≥ 2	1	$\frac{2m-2}{m}$
≥ 2	≥ 2	$\frac{4nm-2n-2m}{nm}$

- The girth of $G_{n,m}$ is 4 for $n, m \geq 3$ and 0 or 1 in other cases.
- The diameter of $G_{n,m}$ is $m + n - 2$.

(iii)

- for $v \in V_i$, $d(v) = \sum_{j \neq i} |V_j| = n - n_i$; average degree : $\sum_{i=1}^r \frac{n_i}{n} (1 - \frac{n_i}{n})$
- $|E(K_{n_1, \dots, n_r})| = \frac{1}{2} \sum_{v \in V} d(v) = \frac{1}{2} \sum_{i=1}^r n_i (n - n_i)$
- girth = 3 if $r \geq 3$ (there are triangles everywhere), girth = 4 if $r = 2$

- diameter = 2

Exercise 3

[Adjacency Matrix]

- (i) Let $A^2 = (b_{ij})$. Theorem 1.6 states that b_{ii} counts the number of paths of length 2 from vertex v_i to itself. Each such path $v_i v_j v_i$ corresponds to an edge $v_i v_j \in E(G)$ for $v_j \in V(G)$, and conversely, each edge $v_i v_j \in E(G)$ corresponds to exactly two paths; namely $v_i v_j v_i$ and $v_j v_i v_j$. Hence, $\text{trace}(A^2) = \sum_{i=1}^{|V(G)|} b_{ii} = 2|E(G)|$.
- (ii) In general, $\text{trace}(A^k)$ counts the number of closed walks in G of length k (i.e. sequences of k edges, which starts and finishes at the same vertex). If a closed walk is of length 3, then it is necessarily a triangle, because a walk using twice the same edge cannot finish where it started. So A_{ii}^3 counts twice the number of triangles containing the vertex i ("twice" because each triangle can be walked on in two directions). So $\sum_i A_{ii}^3$ counts twice the number of pairs (t, i) where t is a triangle of G , and i is a vertex of t . There are 3 vertices in a triangle, so $\sum_i A_{ii}^3 = 2\#\{(t, i) ; t \text{ triangle of } G, i \in t\} = 2 \times 3\#\{t ; t \text{ triangle of } G\} = 6 \times (\text{number of triangles of } G)$.
- (iii) Let $B^3 = (b_{ij})$. All paths of length 3 from a vertex v_i to itself are of the form $v_i w z v_i$ for all $\{w, z\} \in \binom{V(G) \setminus \{v_i\}}{2}$, and each subset $\{w, z\} \in \binom{V(G) \setminus \{v_i\}}{2}$ corresponds to two paths $v_i w z v_i$ and $v_i z w v_i$. Therefore,

$$\begin{aligned} \text{trace}(B^3) &= \sum_{i=1}^n b_{ii} = \sum_{i=1}^n 2 \binom{n-1}{2} \\ &= n(n-1)(n-2) \end{aligned}$$

- (iv) Let $B^4 = (b_{ij})$. There are 4 different types of paths from a vertex v_i to itself of length 4:
- $v_i x y z v_i$. These correspond to subsets $\{x, y, z\} \in \binom{[n] \setminus \{v_i\}}{3}$, and each subset $\{x, y, z\} \in \binom{[n] \setminus \{v_i\}}{3}$ corresponds to 6 paths.
 - $v_i x y x v_i$. These correspond to subsets $\{x, y\} \in \binom{[n] \setminus \{v_i\}}{2}$, and each subset $\{x, y\} \in \binom{[n] \setminus \{v_i\}}{2}$ corresponds to 2 paths.
 - $v_i x v_i y v_i$. These correspond to subsets $\{x, y\} \in \binom{[n] \setminus \{v_i\}}{2}$, and each subset $\{x, y\} \in \binom{[n] \setminus \{v_i\}}{2}$ corresponds to 2 paths.
 - $v_i x v_i x v_i$. These correspond to elements $x \in V(G) \setminus \{v_i\}$, and each element $x \in V(G) \setminus \{v_i\}$ corresponds to a unique path.

Thus, we have:

$$\begin{aligned} \text{trace}(B^4) &= \sum_{i=1}^n b_{ii} = n \left(6 \binom{n-1}{3} + 2 \binom{n-1}{2} + 2 \binom{n-1}{2} + (n-1) \right) \\ &= n^4 + 2n^3 + n^2. \end{aligned}$$

Exercise 4

[Bridges]

- (i) Removing any edge of a tree creates 2 connected component : all edges of trees are bridges.
- (ii) If we remove any edge from $K_{m,n}$, it remains connected if and only if every vertex has degree at least 2, i.e., if $m, n \geq 2$. The only complete bipartite graphs with a bridge are $K_{1,n}$ (or $K_{m,1}$).
- (iii) Without loss of generality, assume that G is connected. Suppose e lies on a cycle $C = u_1 \dots u_n u_1$ of G and let $e = u_n u_1$ without loss of generality. Let u, v be vertices of $G \setminus e$. Then there exists a $u-v$ path P in G , since G is connected. If P does not use edge e , then it is a $u-v$ path in $G \setminus e$. Suppose P uses edge e . Replacing e in P with the path $u_1 \dots u_n$ gives a path from u to v in $G \setminus e$ that does not use e . Hence, there exists a $u-v$ path in $G \setminus e$, and the number of components does not increase. Thus, e is not a bridge.

Now suppose $e = uv$ is not on any cycle. Then u and v lie in different components of $G \setminus e$. For if they were in the same component, then there would be a $u-v$ path in $G \setminus e$ and hence also in G . Together with e , this path would then form a cycle, contradicting the assumption. Thus, $G \setminus e$ has at least two components, and e is a bridge.

Exercise 5

[Bonus – Graph of friendship]

Let $G = (V, E)$ be the graph where V is the set of people present at the party, and there is an edge between two people if they know each other. We want to prove that there exists two vertices of G with the same degree. Let $n = |V|$, then degrees are between 0 and $n-1$, so $d : V \rightarrow \{0, \dots, n-1\}$. If d is injective, then its image is of cardinal $|V| = n$, so its image is exactly $\{0, \dots, n-1\}$. This implies there is a vertex u with degree 0 (someone that knows no-one), and a vertex v with degree $n-1$ (someone that knows everybody). This is clearly not possible: is $(u, v) \in E$ or not ? Hence d is not injective: there exist $u, v \in V$ with $u \neq v$ and $d(u) = d(v)$.

Exercise 6

[With ChatGPT]

Apply the formula claimed by ChatGPT to a triangle (i.e. the complete graph on 3 vertices): it has 3 vertices of degree 2, and 1 triangle. The claimed formula

gives $2 \cdot 3 \cdot 2^2 + 4 \cdot 1 + 0 = 28$. But the adjacency matrix is $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$,

with $A^4 = \begin{pmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{pmatrix}$, so $\text{trace}(A^4) = 18 \neq 28$.

Its error comes from the total bullshit it is saying about triangles.

Exercise 7

[König's lemma and bipartite graphs]

1. If $G = (V, E)$ is bipartite, then let $V = A \sqcup B$ such that there is no edge in between the vertices in A nor in between the vertices in B . Then coloring A in white and B in black gives a 2-coloration of G . Reciprocally, if G is 2-colorable, then let A be the set of white vertices, and B the black ones. We have $V = A \sqcup B$, and there is no edge in between the vertices in A , nor in B . Hence, G is bipartite.
2. If G is k -colorable, then so is any of its sub-graphs (by direct application of definitions). An odd cycle is not 2-colorable because it is not bipartite, hence a 2-colorable graph has no odd cycle.
3. (a) A graph is bipartite if and only if all its connected component are. A graph is odd-cycle-free if and only if all its connected component are. Hence we can restrict to connected component.
 (b) Suppose there is $y \in V_{\leq 2p}$ and $z \in V_{\leq 2q}$ such that $(y, z) \in E$. Consider a path u_0, \dots, u_{2p} with $u_0 = x$ and $u_{2p} = y$, and a path v_0, \dots, v_{2q} with $v_0 = x$ and $v_{2q} = z$. There is an edge $(u_{2p}, v_{2q}) \in E$, hence $\mathcal{P} = (u_0, u_1, \dots, u_{2p}, v_{2q}, v_{2q-1}, \dots, v_0)$ is an edge-path in G that goes from x to x . It has length $2p + 2q + 1$, which is odd, but it is not necessarily a cycle because some vertices can be the same (i.e. $u_i = v_j$ for some i, j). Consider the union of all the edges in \mathcal{P} , and remove the edges that appear twice to obtain \mathcal{Q} . This gives a collection of cycles, and \mathcal{Q} contains a odd number of edges (because we have removed an even number of edges from \mathcal{P}). Consequently, at least one of the cycles in \mathcal{Q} is odd.
 (c) Suppose G has no odd cycle and fix $x \in V$. Color x in white, and all the vertices at even distance of x in white, and the vertices at odd distance in black. This is a 2-coloration of G because (i) there is no edge between white vertices (otherwise there will be an odd cycle according to the previous question), and (ii) there is no edge between black vertices (apply the lemma of the last question from a black vertex instead of x). Thus if G has no odd cycle, it is 2-colorable, which means it is bipartite.

Exercise 8

[Complete graph, Path, Cycle, Circulants]

- (1) $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$; degrees = $n - 1$ = average degree ; girth = 3 ; diameter = 1.
- (2) $|E(P_n)| = n - 1$; degrees $\in \{1, 2\}$; average degree = $\frac{2n-2}{n}$; girth = $+\infty$; diameter = n .
- (3) $|E(C_n)| = n$; degrees = 2 = average degree ; girth = n ; diameter = $\lfloor \frac{n}{2} \rfloor$.
- (4) First, assume that $a_\ell < \frac{n}{2}$. For $v_i \in V(C_n(a_1, \dots, a_\ell))$, we have $N(v_i) = \{v_{i \pm a_j} : j \in [\ell]\}$. Since $a_j < \frac{n}{2}$, the elements of this set are pairwise distinct,

and thus $d(v_i) = 2\ell$. It follows that

$$|E| = \frac{1}{2} \sum_{v \in V} d(v) = \frac{1}{2} n \cdot 2\ell = n\ell.$$

If $a_\ell = \frac{n}{2}$, then $d(v) = 2\ell - 1$ for every $v \in V$, and therefore

$$|E| = \frac{2n\ell - n}{2}$$

Exercise 9

[Mantel's theorem]

1. The number of edges of $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is $\lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$. Hence Mantel's inequality is optimal because the number of vertices of $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is n .
2. Take two disjoint sub-set of V with no edge between them (not necessarily connected components), say on V_1 and V_2 with $|V_1| = n_1$ and $|V_2| = n_2$. Both sub-graphs are triangle-free, so $|E_1| \leq \frac{n_1^2}{4}$ and $|E_2| \leq \frac{n_2^2}{4}$. Consequently, as there is no edge between V_1 and V_2 : $|E| = |E_1| + |E_2| \leq \frac{n_1^2}{4} + \frac{n_2^2}{4} \leq \left(\frac{n_1+n_2}{2}\right)^2 = \frac{(n_1+n_2)^2}{4}$. By induction, if Mantel's inequality hold for connected components, it holds for the whole graph.
3. As G has no triangle, if x is a neighbor of u , then it is **not** a neighbor of v . Thus, each $x \in V \setminus \{u, v\}$ share an edge with at most one of the vertices u or v : the number of edges between $\{u, v\}$ and $V \setminus \{u, v\}$ is at most $|V \setminus \{u, v\}| = n - 1$. Adding the edge (u, v) , we get $d(u) + d(v) \leq n$.
4. Our induction is going to remove 2 vertices at each step, so we initialize with $n = 1$ and $n = 2$. For $n = 1$, there is no edge, so the statement holds; for $n = 2$, there is at most 1 edge, so the statement holds. Suppose Mantel's theorem holds for all graphs on $n - 2$ vertices, and fix $G = (V, E)$ on n vertices. Pick an edge $(u, v) \in E$, and consider the graph H obtained by removing u and v from G . The graph H has $n - 2$ vertices, so $|E(H)| \leq \frac{(n-2)^2}{4}$. Besides, edges of G are either contained in H or adjacent to u or v , so there are (the -1 avoids the double counting of the edge (u, v)):

$$|E| = |E(H)| + d(u) + d(v) - 1 \leq \frac{(n-2)^2}{4} + n - 1 = \frac{n^2}{4}$$

One can actually prove that the balanced bipartite graph of 1. is the only graph without triangle to achieve equality in Mantel's theorem, but we will not do that here.

Exercise 10

[Connectivity when discarding a vertex]

1. Complete graph
2. Path
3. (a) t and t' are also vertices of G , and G is connected.
 - (b) If $s_1 \neq t_j$, then all t_j are vertices of H , hence (t_1, \dots, t_m) is a path in H .
 - (c) If $t_{j-1} \notin \{s_2, \dots, s_r\}$, then one can add t_{j-1} to the path to obtain $(t_{j-1}, s_1, \dots, s_r)$ because $t_{j-1}s_1$ is an edge of G . Then this new path is longer than (s_1, \dots, s_r) , contradicting the assumption. The same holds for t_{j+1} .
 - (d) Assume $u \leq v$, and consider $(t_1, \dots, t_{j-1}, s_{u+1}, \dots, s_{v-1}, t_{j+1}, \dots, t_m)$. All vertices are in H by construction, and there is an edge between every consecutive vertices, and t and t' are connected in H (the same holds if $v \leq u$). Consequently, H is connected (we have proven that every two vertices are connected).
 - (e) Actually, we have proven that every end point of a longest path can be discarded without infringing the connectivity of G . This is the case for s_r also, and $s_r \neq s_1$, hence for every graph there are at least 2 vertices that can be discarded without breaking connectivity. For a path, there are **exactly** 2 such vertices.

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 3

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1

[Perfect matching]

A *perfect matching* M in a graph $G = (V, E)$ is a collection of edges $M \subseteq E$ such that every vertex in V is adjacent to **exactly one** of the edges of M .

1. Show that K_n has a perfect matching if and only if n is even.
2. Show that $K_{n,m}$ has a perfect matching if and only if $n = m$.
3. How many different perfect matchings has $K_{3,3}$?
4. Alice and Bob play a game against each other on a graph G . Alice starts by choosing a vertex, then each player plays alternatively by choosing a neighbor of the last vertex chosen (that have not been chosen before). The first player that cannot play loses. Show that if G has a perfect matching, then Bob can always win, whatever Alice does.

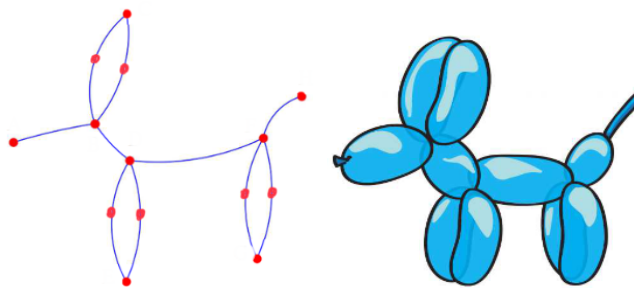


Abbildung 3: A balloon sculpture

Exercise 2

[Trail partition and balloon sculptures]

You need to help a balloon artist to make balloon sculptures, see Figure 3 (Right). A graph $G = (V, E)$ can be made of k balloons if there exists k disjoint edge-paths P_1, \dots, P_k such that every edge $e \in E$ belongs to exactly one of the path P_i (an edge-path can use a vertex several times; two edge-paths are disjoint when they do not share any common edge, they could share vertices).

1. The graph of Figure 3 (Left) can be made of 1 balloon, as illustrated in Figure 3 (Right). Which theorem of the lecture ensures that ?
2. Suppose a connected graph G can be made of $k \geq 1$ balloons. Show that G has an even number m of vertices of odd degree, with $m \leq 2k$.

3. How many balloons do you need to make the Platonic solids whose graphs are given in Figure 4? For each, draw the partition of the graph made by these balloons, as illustrated in Figure 4 (second graph).

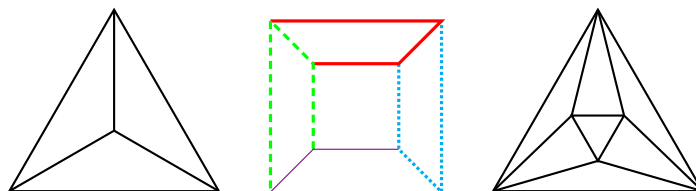


Abbildung 4: Graphs of some Platonic solids. The second graph is already decomposed into 4 balloons (= 1 blue dotted + 1 red thick + 1 green dashed + 1 violet thin); you have to decompose the other graphs into balloons.

Exercise 3

[Degree Sequences in Trees]

1. Let d_1, \dots, d_n be positive natural numbers with $n \geq 2$ and $\sum_{i=1}^n d_i = 2n - 2$. Show that there exists a tree on n vertices with vertex degrees d_1, \dots, d_n . The sequence d_1, \dots, d_n is also referred to as a degree sequence.
2. Let T be a tree with 12 vertices, with exactly three vertices of degree 3 and exactly one vertex of degree 2. Determine the degree sequence of T .
3. Does the degree sequence in (2.) uniquely define the tree T (up to isomorphism)? If not, draw two non-isomorphic trees with this degree sequence.

Exercise 4

[Making non-Eulerian graphs Eulerian]

1. Prove that the number of odd-degree vertices of a G is always even.
2. Let G be a graph which is not Eulerian. Prove that G can be made Eulerian by adding 1 vertex and some edges.
3. Let G be a graph which is not path-Eulerian. State and prove a necessary and sufficient condition on G such that there exists a path-Eulerian graph H obtained by *adding* 1 vertex and some edges to G .
4. Let G be a graph which is not Eulerian, with $2k$ odd-degree vertices. State and prove a necessary and sufficient condition on G such that there exists a Eulerian graph H obtained by *removing* exactly k edges to G . *Hint:* Look at the graph formed by the vertices of G of odd degree and the edges between them, then re-read Exercise 1.

Exercise 5

[Bonus – Organizing a school trip]

Every day, a group of 12 children go for a walk, holding hands by pairs (like you probably did a long time ago). How many days can they go for a walk, if we don't want any child to have the same neighbor twice? What if the walk is now done in rows of three?

Submission: by 14.05.2025 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

ChatGPT told me this (it is a real copy-paste from the free version of ChatGPT):

“ We are asked: How many bipartite graphs are there on 3 labeled vertices?

A graph is bipartite if its vertex set can be partitioned into two disjoint subsets A and B , with edges only between A and B .

Step 1: Enumerate all bipartitions The possible bipartitions of $V = \{1, 2, 3\}$ are:

- $|A| = 1, |B| = 2$: There are $\binom{3}{1} = 3$ such partitions. For each, there are $2^2 = 4$ possible bipartite graphs. - $|A| = 2, |B| = 1$: Again, $\binom{3}{2} = 3$, and each gives $2^2 = 4$ bipartite graphs.

Thus, the total number of bipartite graphs is:

$$3 \cdot 4 + 3 \cdot 4 = 24.$$

Including the empty graph (which corresponds to the case where no edges are present), we obtain the total number:

$$24 + 1 = 25.$$

Thus, there are 25 bipartite graphs on 3 labeled vertices. ”

This is false, why?

Exercise 7

[Number of perfect matchings]

1. How many perfect matchings are there in the path P_n (distinguish n odd or even)?
2. How many perfect matchings are there in the cycle C_n (distinguish n odd or even)?
3. How many perfect matchings are there in the complete graph K_n (distinguish n odd or even)?
4. How many perfect matchings are there in the complete bipartite graph $K_{n,n}$?

Exercise 8

[Fleury's algorithm]

In the lecture, you have seen a proof of the following theorem: *if a graph has 2 vertices of odd degree, then it has an Eulerian trail*. From the lecture, one can retrieve an algorithm (Hierholzer's algorithm, 1873) that explicitly construct the Eulerian trail. Here, we give another algorithm (which is easier to prove, but less efficient in practice): Fleury's algorithm (1883).

- (a) Fix a graph G . Start at a u vertex of odd degree, and an empty path \mathcal{P} .
- (b) Choose a neighbor of u such that $G \setminus uv$ is connected, and add the edge uv to your path \mathcal{P} .
- (c) If no such neighbor exists, then u shall have only 1 adjacent edge: add it to your path \mathcal{P} and terminate.

- (d) Remove the edge uv from your graph, and repeat the process from v (in the graph $G \setminus uv$) at step (b).

Prove that this algorithm terminates and that, when it stops, the trail \mathcal{P} is an Eulerian trail. What is the complexity of this algorithm?

Exercise 9

[Adjacency matrix – again]

In what follows, we consider the complete bipartite graph $K_{n,m}$, and its adjacency matrix $A_{n,m}$. The vertices of $K_{n,m}$ are labelled a_1, \dots, a_n and b_1, \dots, b_m such that its edges are $a_i b_j$ for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. We label the coordinates of $\mathbf{v} \in \mathbb{R}^{n+m}$ accordingly.

1. Prove that 0 is an eigenvalue of multiplicity at least $n + m - 2$ of $A_{n,m}$.
(Hint : search an eigenvector with very few non-zero coordinates.)
2. For $n = m$, prove that n and $-n$ are eigenvalues of $A_{n,n}$. Deduce its diagonalization (i.e. give all the eigenvectors together with the eigenvalues).
3. For $n \neq m$, and $\mathbf{v} \in \mathbb{R}^{n+m}$, prove that the value of the coordinate of $A\mathbf{v}$ on the vertex a_i does not depend on i .
4. For $n \neq m$, find the eigenvectors of $A_{n,m}$ associated to a non-zero eigenvalue, and deduce it is diagonalizable.

Exercise 10

[Exercise 4 – continuation]

If you have solved questions 1 and 2 of the above Exercise 4 and the exercises on adjacency matrix from the previous Übungsbätter:

5. Prove that there are n^{k-1} cycles of even length in $K_{n,n}$ that contains a_1 (here, cycles are allowed to use several times the same edge), and 0 cycle of odd length.

Exercise 11

[Dominoes]

You are given a pile of dominoes, containing all possible dominoes you can make using numbers from 1 to 5. As per usual, two dominoes can be put side to side if their end matches. How many dominoes do you have? Can you arrange your dominoes into a circle using all your dominoes?

Replace 5 by n , and re-do the exercise.

Exercise 12

[Wire-frame of a cube]

You have 120cm of steel wire. You want to construct (the frame of) a cube with sides of 10cm. Can you do it?

Exercise 13

[Polygamy Matching Theorem and Tic-Tac-Toe]

1. Let $G = (S \cup T, E)$ be a bipartite graph such that for all $A \subseteq S$, the *polygamy matching condition* $|N(A)| \geq 2 \cdot |A|$ holds. Show that there exists a family of pairwise disjoint subgraphs of G that are isomorphic to $K_{1,2}$, such that each node from S is the center of one of these subgraphs $K_{1,2}$. (Hint: Consider the auxiliary graph G' , where for each node $v \in S$, another node v' is inserted, which has the same neighbors as v . Apply the standard matching theorem to G' .)

2. A *positional game* consists of a *game board*, modeled by the set $X = \{x_1, \dots, x_n\}$ of squares, and so-called *winning sets*, modeled by subsets $W_1, \dots, W_m \subset X$. For example, Tic-Tac-Toe is a positional game with a game board of 9 squares and 8 winning sets (the vertical, horizontal, and diagonal lines). Two players take turns selecting squares from X . The player who first selects all squares from a winning set wins. Consider a positional game with the property that each winning set consists of at least 10 squares, and each square of the game board lies in at most 5 winning sets. Show that the second player can always achieve at least a *draw*. (Hint: Show there exists a set of disjoint pairs of squares such that each winning set contains at least one of these pairs, and use it to determine a strategy for the second player.)

Discrete Mathematics – Solutions Sheet 3

Exercise 1

[Perfect matchings]

1. A matching covers an even number of vertices (that is $2 \# M$), so if n is odd, there exists no matching that covers every vertex. If n is even, then $M = \{(2i)(2i+1) ; i \in \{1, \dots, \frac{n}{2}\}\}$ is a perfect matching (you shall draw it).
2. The edges in a matching in $K_{n,m} = (A \cup B, E)$ select a left vertex (in A) and a right vertex (in B), hence if M is perfect, then every vertex in A is *matched* with a vertex of B , that is to say $n = |A| = |B| = m$. Conversely, if $n = m$, then write $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, and take $M = \{a_i b_i ; i \in \{1, \dots, n\}\}$.
3. A perfect matching of $K_{n,n}$ is equivalent to a bijection σ of $\{1, \dots, n\}$ by setting $\sigma(i) = j$ for $ij \in M$. Hence, there are the number of perfect matchings as bijections of $\{1, \dots, n\}$, i.e. permutation: there are $n!$ perfect matchings of $K_{n,n}$. For $n = 3$, there are $3! = 6$ perfect matchings of $K_{3,3}$ (you shall draw them).
4. Fix a perfect matching M . Bob strategy is the following: when Alice plays the vertex u , there exists a (unique) vertex v_u such that uv_u is an edge of the matching M (whatever u , as M is perfect), then Bob plays v_u . We need to prove that it is always possible for Bob to play, whatever vertex Alice chooses, that is to say prove that v_u has not been played before (v_u is indeed a neighbor of u in G because the matching ensures it). Suppose that is not the case: v_u has been played before. Can Bob have played it? No because Bob only plays edges from the matching, so Bob plays v_u only when u is played (and u has not been played before). Can Alice have played it? No because if Alice plays v_u , then Bob plays u (and it is Alice's turn), and u has not been played before. Hence, whatever Alice plays, Bob can answer, and as the graph is finite, Alice will run out of possibilities and lose at some point.

Exercise 2

[Trail partition and balloon sculpture]

1. 1 balloon is equivalent to an Eulerian edge-path. Hence, being made of 1 balloon is equivalent to being path-Eulerian. According to the lecture, this is equivalent to having exactly 2 vertices of odd degree. As this is the case of the dog graph, it can be made of 1 balloon.
2. Each balloon is a sub-graph of G with exactly 2 vertices of odd degree. The degree of a vertex v of G is the sum of the degree of v in each balloon, so if its degree is odd, then it has an odd degree in at least 1 balloon (if it

were even in every balloon, then it will be even in G). Hence, the number of vertices with odd degree can not exceed the number of endpoints of balloons: $m \leq 2k$. Note that the converse is true but more complicated to show: if G has $2k$ vertices of odd degree, then it can be made of k balloons.

3. See Figure 5. The tetrahedron (left) can be made of 2 balloons: it can not be made of 1 balloons because it has > 2 odd vertices, and the figure show how to do it with 2 balloons. The octahedron (right) is Eulerian because it has no odd vertices, the figure indicates how to order edges.

You can do the exercise on the graphs of the two other Platonic solids, given below.

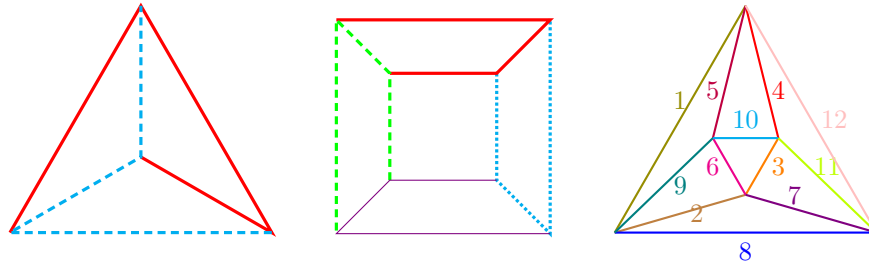


Abbildung 5: Solution to balloon problem

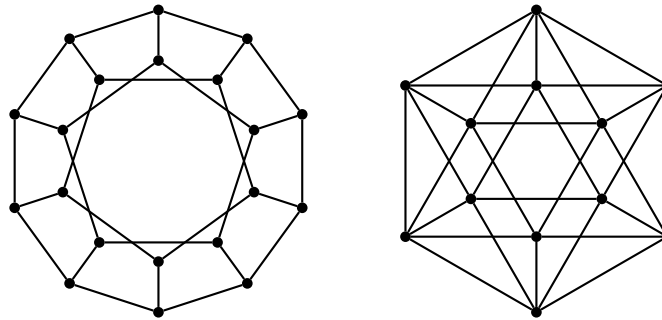


Abbildung 6: The dodecahedral graph and the icosahedral graph. (Good luck!)

Exercise 3

[Degree sequences in Trees]

- (i) Let d_1, \dots, d_n be given with $\sum_{i=1}^n d_i = 2n - 2$. We prove the assertion by induction on n . If $n = 2$, then $(1, 1)$ is the only possible sequence that satisfies the conditions, and the tree with 2 nodes and one edge has exactly this degree sequence. Let $n \geq 3$, there exists an i such that $d_i < 2$. Otherwise, $\sum_{i=1}^n d_i \geq 2n$ would hold. Since $d_i > 0$ by assumption, we have $d_i = 1$. On the other hand, there must also exist a j with $d_j > 1$ (otherwise $\sum_{k=1}^n d_k \leq n$). Without loss of generality, let $d_n = 1$ and

$d_{n-1} > 1$. By the induction hypothesis, there exists a tree with degree sequence $d_1, \dots, d_{n-2}, d_{n-1} - 1$. Adding a new leaf to the node $n - 1$ yields a tree with the degree sequence d_1, \dots, d_n .

- (ii) Let $V(T) = \{u_1, \dots, u_{12}\}$ and let $d(u_{12}) = 2, d(u_9) = d(u_{10}) = d(u_{11}) = 3$. According to the lecture,

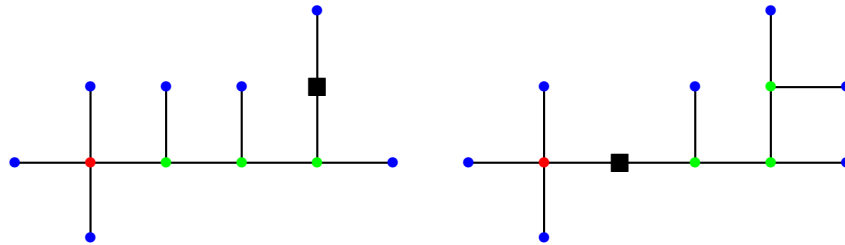
$$\sum_{i=1}^{12} d(u_i) = 2 \cdot |V(T)| - 2 = 22.$$

Since $\sum_{i=9}^{12} d(u_i) = 11$ by assumption, it follows that

$$\sum_{i=1}^8 d(u_i) = 11.$$

Furthermore, we know that $d(u_i) \in \{1\} \cup \{a : a \geq 4\}$. Therefore, there must exist a node u_i with $d(u_i) \geq 4$ (otherwise $\sum_{i=1}^8 d(u_i) \leq 8$). On the other hand, there cannot be a node with degree at least 5, otherwise $\sum_{i=1}^{12} d(u_i) \geq 7 + 5 = 12$. Thus, there exists a node with degree 4 and without loss of generality, let $d(u_8) = 4$. Then $\sum_{i=1}^7 d(u_i) = 7$, and since all degrees are at least 1, we must have $d(u_1) = \dots = d(u_7) = 1$. Therefore, the degree sequence is $(1, 1, 1, 1, 1, 1, 1, 2, 3, 3, 3, 4)$.

- (iii) The degree sequence does not uniquely determine the tree. The following illustration shows two non-isomorphic trees with the same degree sequence (the one from above) (the vertex of degree 2 is adjacent to the vertex of degree 4 in the right tree and not in the left tree).



Exercise 4

[Making non-Eulerian graphs Eulerian]

1. By the hand-shaking lemma: $\sum_{v \in V(G)} d_v = 2 \#E$, so there is an even number of terms which are odd in the sum, i.e. the number of odd-degree vertices of G is even.

2. As G is not Eulerian, it has at least 1 odd-degree vertex. Let u_1, \dots, u_{2k} be the odd-degree vertices of G (but the previous question, we know there are an even number of them). Consider the graph H formed by G together with an additional vertex v_o , and the edges $u_i v_o$ for all $i \in [2k]$. All the vertices of this new graph H are of even degree: the even-degree vertices of G are still of even degree in H ; the odd-degree ones have gained 1 edge; and v_o has degree $2k$. Hence, H is Eulerian.
3. We will prove that: all non(path-Eulerian) graphs G can be made path-Eulerian by adding 1 vertex and some edges. As previously, consider H obtained by adding v_o and all the edges $u_i v_o$ for u_i odd-degree vertex of G . We know that H has 0 odd-degree vertex, we want that it has 2 odd-degree vertices, in order to be path-Eulerian. If G has an even-degree vertex w , add to H an edge $w v_o$, and you are done. If G has an odd-degree vertex u_1 , remove from H the edge $u_1 v_o$, and you are done (note that H is still connected because if G has an odd-degree vertex, then it has at least 2 because the number of odd-degree vertices is even).
4. Let G_{odd} be the graph whose vertex set is the set $\{u_1, \dots, u_{2k}\}$ of odd-degree vertices of G , and whose edges are all the edges of G of the form $u_i u_j$ (this is called the sub-graph of G induced on its odd-degree vertices). We will prove that: G can be made Eulerian by removing k edges if and only if G_{odd} admits a perfect matching. Suppose G_{odd} admits a perfect matching: Removing from G the edges of the perfect matching reduces the degree of each odd-degree vertex of G by 1, hence the resulting graph has only even-degree vertices and is Eulerian (and there are k edges in a perfect matching). Conversely, if G can be made Eulerian by removing k edges, consider the sub-graph G' of G formed by these k edges. Note that for each $i \in [2k]$, the degree of u_i in G' is odd (you need to remove an odd number of edges adjacent to u_i in order to make it of even degree in G). As every odd number is ≥ 1 , and as there are $2k$ odd-degree vertices but only k edges, the degree of u_i in G' is exactly 1. Thus, G' consists of isolated edges: it is a perfect matching between u_1, \dots, u_{2k} (i.e. a perfect matching in G_{odd}). **N.B.:** Note that k is the minimal number of edges one can remove in order to make G Eulerian. In general, if you are allowed to remove more than k edges, then the induced sub-graph G_{odd} only needs to have a sub-graph with no vertex of even-degree, which is far more difficult to check.

Exercise 5

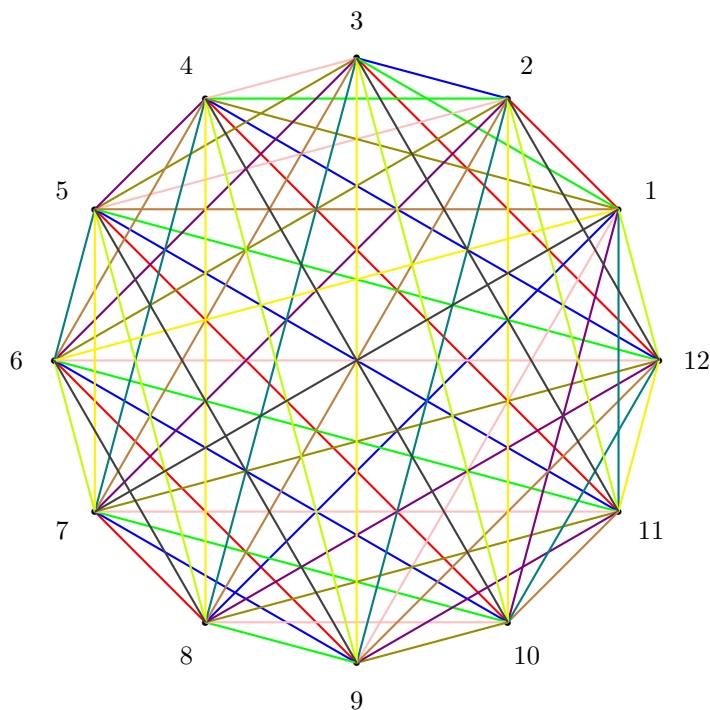
[Bonus – Organizing a school trip]

One day of holding hands by pairs amounts to a perfect matching in the complete graph on 12 vertices. Hence, it uses $\frac{12}{2} = 6$ edges. In the complete graph, there are $\binom{12}{2} = \frac{12 \times 11}{2} = 66$ edges. Hence, if all the edges of all the perfect matchings are different (which is the translation of the condition “we don’t want any child to have the same neighbor twice”), there could be at most $\frac{66}{6} = 11$ perfect

matchings: our trip cannot be more than 11 days long (in general for an even number n of children, it cannot be more than $\frac{2}{n} \binom{n}{2} = n - 1$ days long).

It remains to prove that we can indeed find 11 disjoint perfect matchings in K_{12} . Here is an example (each perfect matching is made of 6 edges):

1 – 2	3 – 12	4 – 11	5 – 10	6 – 9	7 – 8
2 – 3	12 – 4	11 – 5	10 – 6	9 – 7	8 – 1
1 – 3	4 – 2	5 – 12	6 – 11	7 – 10	8 – 9
3 – 4	2 – 5	12 – 6	11 – 7	10 – 8	9 – 1
1 – 4	5 – 3	6 – 2	7 – 12	8 – 11	9 – 10
4 – 5	3 – 6	2 – 7	12 – 8	11 – 9	10 – 1
1 – 5	6 – 4	7 – 3	8 – 2	9 – 12	10 – 11
5 – 6	4 – 7	3 – 8	2 – 9	12 – 10	11 – 1
1 – 6	7 – 5	8 – 4	9 – 3	10 – 2	11 – 12
6 – 7	5 – 8	4 – 9	3 – 10	2 – 11	12 – 1
1 – 7	2 – 12	3 – 11	4 – 10	5 – 9	6 – 8



If the walk is done in rows of three, then there are 4 rows, and each day you use $3 \times 4 = 12$ edges of K_{12} . Thus, your trip can last for at most $\lfloor \frac{66}{12} \rfloor = 5$ days. Good luck for finding a solution (write me a correct one, and I will add it here)!

Exercise 6

[With ChatGPT]

There are only $2^3 = 8$ graphs on 3 (labeled) vertices. How can there be 25 bipartite ones?

Exercise 7

[Number of perfect matchings]

Exercise 8

[Adjacency matrix – again]

What needs to be understood is the action that the adjacency matrix A has on the graph. First, a vector $\mathbf{v} \in \mathbb{R}^{n+m}$ is just a weight on the vertices (i.e. one can draw the graph and write the coordinates of \mathbf{v} directly on the vertices), and $A\mathbf{v}$ is also a vector: the value of $A\mathbf{v}$ on a vertex u is the sum of the values of \mathbf{v} on the neighbors of u .

1. Take $\mathbf{x}^i \in \mathbb{R}^{n+m}$ with coordinated $x_{a_1}^i = 1$ and $x_{a_i}^i = -1$ for $i \neq 1$, and $v_\alpha = 0$ if $\alpha \notin \{a_1, a_i\}$ (especially if $\alpha = b_j$). Then $A_{n,m}\mathbf{x}^i = \mathbf{0}$. This works for all $i \in \{1, \dots, n\}$, and by symmetry, for \mathbf{y}^j with $y_{b_1}^j = -y_{b_j}^j = 1$ and 0 else, for $j \in \{1, \dots, m\}$. We get $(n-1) + (m-1)$ vectors in the kernel of A . All these vectors are linearly independent because they are echelonized (write \mathbf{x}^i and \mathbf{y}^j as columns of a matrix: there is an identity matrix of full rank inside).
2. The graph $K_{n,n}$ is regular, so take $\mathbf{z} \in \mathbb{R}^{2n}$ with $z_{a_i} = z_{b_j} = 1$, then $A\mathbf{z} = n\mathbf{z}$. Take $\mathbf{t} \in \mathbb{R}^{2n}$ with $t_{a_i} = 1$ and $t_{b_j} = -1$, then $A\mathbf{t} = -n\mathbf{t}$. The spectrum of $A_{n,n}$ is $\{-n, 0, +n\}$, the eigenvalues n and $-n$ have multiplicity 1 (their eigenvectors being \mathbf{z} and \mathbf{t} respectively), and the eigenvalue 0 has multiplicity $2n-2$ (an eigenbasis of its eigenspace being given by the vectors \mathbf{x}^i and \mathbf{y}^i for $i \in \{1, \dots, n\}$). As we have found a total multiplicity of $1 + 1 + (2n-2) = 2n$, which is the size of the matrix $A_{n,n}$, we have a diagonalization.
3. The coordinate of $A\mathbf{v}$ on the vertex a_i is the sum of the coordinates of \mathbf{v} on the neighbors of a_i . As $K_{n,m}$ is complete, the neighbors of a_i are (b_1, \dots, b_m) , which does not depend on i . Hence, the coordinate of $A\mathbf{v}$ on the vertex a_i does not depend on i : it is $\sum_j v_{b_j}$.
4. If $\mathbf{v} \in \mathbb{R}^{n+m}$ is an eigenvector of $A_{n,m}$ associated to a non-zero eigenvalue, then $A_{n,m}\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \neq 0$. In particular, all the coordinates of $A_{n,m}\mathbf{v}$ on the vertices a_i are the same (according to the previous question), so as $\lambda \neq 0$, all the coordinates of \mathbf{v} on a_i are the same: we denote it α . Similarly, all the coordinates of \mathbf{v} on b_j are the same: we denote it β . Looking at the coordinate on a_1 in the equality $A_{n,m}\mathbf{v} = \lambda\mathbf{v}$, we get $m\beta = \lambda\alpha$; while looking at the coordinate on b_1 , we get $n\alpha = \lambda\beta$ (remember that we are trying to know if λ, α and β can exist, while n and m are fixed, and $n \neq m, \lambda \neq 0$). If $\alpha = 0$, then $\lambda\beta = 0$, which is not possible ($\lambda \neq 0$ by hypothesis, and if $\beta = 0$, then $\mathbf{v} = \mathbf{0}$ but \mathbf{v} is an eigenvector). Thus, $\lambda = \frac{\beta}{\alpha}m$, so $n\alpha = \lambda\beta$ gives $\frac{n}{m} = \left(\frac{\beta}{\alpha}\right)^2$. Finally, one can check that taking $(\alpha, \beta) = (+\sqrt{n}, +\sqrt{m})$ or taking $(\alpha, \beta) = (-\sqrt{n}, +\sqrt{m})$ gives two different (and linearly independent) eigenvectors with eigenvalue $\lambda = \sqrt{nm}$. The matrix $A_{n,m}$ is diagonalizable because we have found eigenvalues with multiplicity adding to $2n$, the size of the matrix.

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 4

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1

[Hamiltonian Graphs]

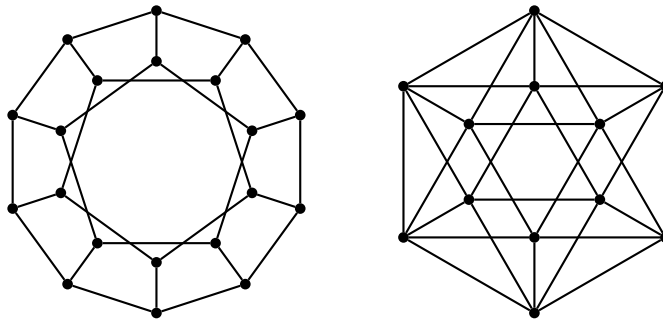


Abbildung 7: The dodecahedral graph and the icosahedral graph.

1. In 1857, Sir William Rowan Hamilton invented the game *The Icosian Game*. The task was to find a Hamiltonian cycle of the graph G in Figure 7 (Left). Show that both graphs of Figure 7 have a Hamiltonian cycle (draw it in color).
2. Let Q_n denote the hypercube graph. Decide for which n the graph Q_n is Eulerian. Decide for which n the graph Q_n is Hamiltonian. Recall that: The hypercube graph Q_n has vertices all the subsets of $\{1, \dots, n\}$, and the edges are between X and Y if $X \subseteq Y$ and $|Y| = |X| + 1$.

Exercise 2

[Optimality of Dirac's Theorem]

1. Construct a graph $G = ([5], E)$ with $d(i) \geq 2$ for all $i \in [5]$, which does not contain a Hamiltonian cycle.
2. Construct, for any $n \in \mathbb{N}$, a graph with n vertices and $1 + \binom{n-1}{2}$ edges that is not Hamiltonian.
3. Construct, for any $n \in \mathbb{N}$, a graph with n vertices and minimum degree $\lceil \frac{n}{2} \rceil - 1$ that is not Hamiltonian.

Exercise 3

[Ore's theorem]

We want to prove the following theorem (Ore, 1960): *if, for every non-adjacent vertices u and w of G we have $d(u) + d(w) \geq n$, then G is Hamiltonian*, where $d(u)$ is the degree of u in G .

1. Prove that the reciprocal of Ore's theorem is false. Hint: 5 vertices are enough.
2. Write the contrapositive of Ore's theorem. This is what we will prove.
3. Let $H = (V, E)$ be a maximally non-Hamiltonian graph on n vertices (i.e. adding any edge to H would create a Hamiltonian cycle).
 - (a) If u and w are not adjacent in H , prove that there is a Hamiltonian path (v_1, v_2, \dots, v_n) with $v_1 = u$ and $v_n = w$.
 - (b) For $i \in \{2, \dots, n-2\}$, prove that if $uv_{i+1} \in E$, then $v_i w \notin E$.
 - (c) Deduce that $d(u) + d(w) \leq n-1$.
4. Prove Ore's theorem (by proving its contrapositive): take G and add edges to it to get a maximally non-Hamiltonian graph H , and conclude.

Exercise 4

[Chinese postman problem]

Kwan Mei-Ko in 1960, asked the following problem: *in the city of Guan, a postman needs to deliver mails to citizens in every streets; how to minimize the number of street in which he needs to pass twice?* (this is not a quote but a reformulation). A city can be seen as a graph whose vertices are the crossroads (intersections between streets, dead-ends are not taken into account), and the edges are between the crossroads which belong to the same street. The postman delivers mails by following a trail (in this graph) which comes back at its starting point.

1. A city is a *US city* when every two intersecting streets meet at a 90° angle, and there is no dead-end. Show that in US cities, the postman does not need to pass by any street twice.
2. Suppose Osnabrück city has only 2 crossroads of odd degree in its graph $G = (V, E)$, namely u and v . Let d be the distance between u and v . Show that the postman can deliver the mails by using twice k edges, with $k \leq d$.
3. We want to show that $k = d$ in Osnabrück. Consider a possible trail of the postman, let E' be the edges he uses twice, and $G' = (V, E')$ the sub-graph obtained.
 - (a) Show that for any $w \in V$ with $w \notin \{u, v\}$, the degree $d'(w)$ of w in G' is even.
 - (b) Suppose u and v are not in the same connected component of G' , use the degree formula to obtain a contradiction.
 - (c) Deduce that $|E'| \geq d$ and conclude.

Exercise 5

[Bonus – Playing chess]

Can a Rook go through all squares of a 8×8 chessboard exactly once and come back to its starting point? If yes, draw how.

Can the King or the Queen do it? Can a Bishop starting on a black square go through all black squares exactly once? Can a Knight go through all squares of a 5×5 chessboard exactly once and come back to its starting point?

Submission: 21.05.2024 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

ChatGPT told me this (it is a real copy-paste from the free version of ChatGPT):

“ We construct a simple graph G on $n = 6$ vertices such that:

- G has **no Hamiltonian cycle**,
- G has two **non-adjacent vertices** u, v with $\deg(u) + \deg(v) \geq 6$,
- G has the **minimal possible number of edges** among such graphs.

Construction The graph has 6 vertices and 7 edges. The edges are:

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{2, 4\}, \{3, 5\}$$

Vertices 2 and 5 are non-adjacent and satisfy $\deg(2) = 3$, $\deg(5) = 3$, so their degree sum is 6.

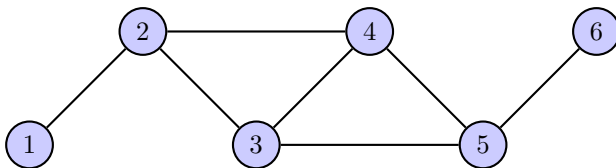


Abbildung 8: A graph with no Hamiltonian cycle, satisfying $\deg(2) + \deg(5) = 6$

Justification The graph contains no Hamiltonian cycle: any attempt to include all vertices in a cycle fails due to forced backtracking or revisiting a vertex.

Moreover, the non-adjacent pair $(2, 5)$ satisfies the degree sum condition: $\deg(2) + \deg(5) = 3 + 3 = 6 = n$.

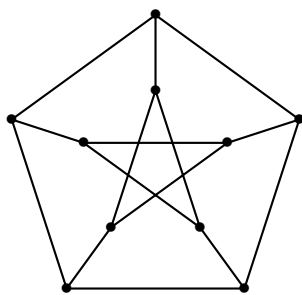
The graph has only 7 edges, which is minimal among all such graphs satisfying the condition. ”

This is false, why?

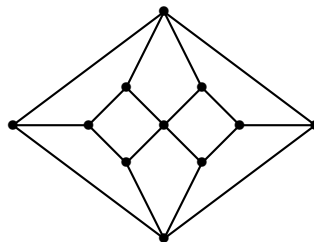
Exercise 7

[Who is Hamiltonian?]

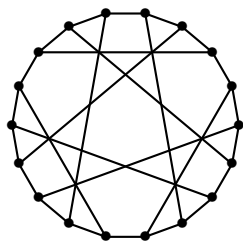
Which of the following graph is Hamiltonian, which is Eulerian, which has an Hamiltonian path, which has an Eulerian path?



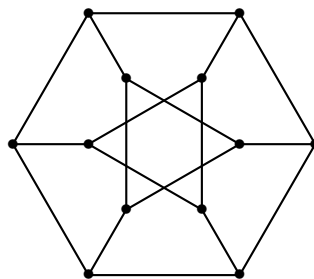
Petersen graph



Herschel graph



Pappus graph



Dürer graph

Exercise 8

[Eulerian vs Hamiltonian]

For each following sentence, give an example or show that such a graph can not exist:

1. A graph that is Eulerian but not Hamiltonian.
2. A graph that is Hamiltonian but not Eulerian.
3. A graph that has an Eulerian path but no Hamiltonian path.
4. A graph that has a Hamiltonian path but no Eulerian path.
5. An Hamiltonian graph with at least 3 vertices, and a bridge.
6. A graph with at least 3 vertices, a bridge, and an Hamiltonian path.

Exercise 9

[Line graph]

For a graph $G = (V, E)$, its *line graph* $L(G)$ is the graph whose vertex set is E and whose edges are between edges of G that share a common vertex.

1. For some graphs of your choice, draw the corresponding line graph (e.g. path, cycles, complete graph, bipartite complete graph, etc). Find a graph G such that $G \neq L(L(G))$, and a graph G with $L(G) = G$.
2. Show that if G is Eulerian, then $L(G)$ is Hamiltonian.
3. For an edge uv of G , let e_{uv} be the corresponding vertex of $L(G)$. Express $d_{L(G)}(e_{uv})$ in term of $d_G(u)$ and $d_G(v)$. Recall: $d_H(x)$ is the degree in the graph H of the vertex x .
4. Show that if G is Eulerian, then $L(G)$ is Eulerian.

5. Give an example of a graph G which is not Hamiltonian nor Eulerian, but for which $L(G)$ is Eulerian.

Exercise 10

[Partitioning into Hamiltonian cycles]

Let G be the graph with vertex set $\{1, \dots, n\}$ and edges uv if $|u - v| \in \{1, 2, n - 1, n - 2\}$ (called the *circulant graph* $Circ(n; 1, 2)$). Show that the vertices of G can be partitioned into 2 Hamiltonian cycles.

Exercise 11

[Product graph]

For two graph $G = (V, E)$ and $G' = (V', E')$, the product $G \square G'$ is the graph with vertex set $V \times V' = \{(v, v') ; v \in V, v' \in V'\}$ and edges $(u, u')(v, v')$ if $(u = v \text{ and } u'v' \in E') \text{ or } (uv \in E \text{ and } u' = v')$.

1. How many edges are there in $G \square G'$
2. Show that $K_2 \square K_2 \square \dots \square K_2$ is isomorphic to Q_n (there are n copies of K_2).
3. Show that there exists G and H , both Eulerian, such that $G \square H$ is not Eulerian.
4. Show that if G and H are Hamiltonian, then $G \square H$ is Hamiltonian.
5. Let T be a tree. Show that if $G \square T$ is Hamiltonian, then $|V(G)| \leq \Delta(T)$ (*recall*: $\Delta(T)$ is the maximum degree of T).

Discrete Mathematics – Solutions Sheet 4

Exercise 1

[Hamiltonian graphs]

1. Figure 9 shows a Hamiltonian cycle in both graphs

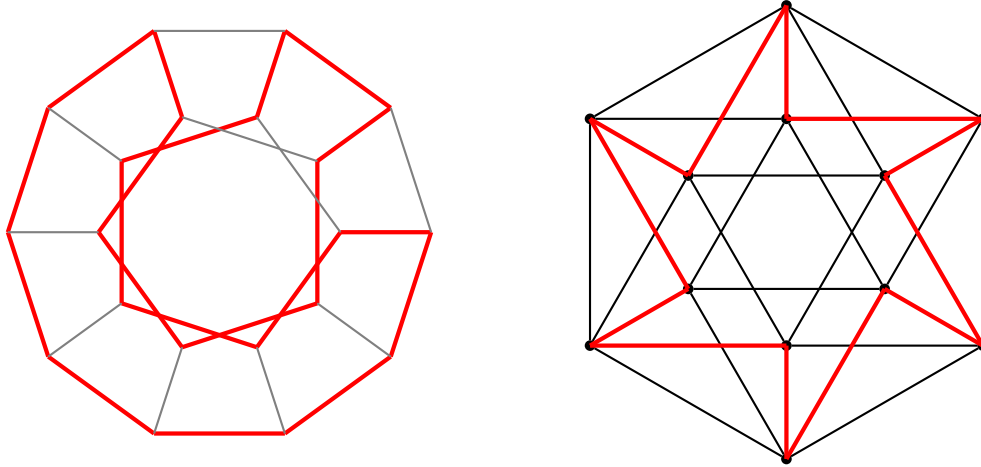


Abbildung 9: Exercise 1(i)

2.
 - The hypercube graph Q_n is a regular graph with $d(v) = n$ for all $v \in V$. From the theorem on Eulerian cycles in the lecture, it follows that a connected graph G is Eulerian if and only if every vertex of G has even degree. Therefore, Q_n is Eulerian if and only if n is even.
 - The hypercube $Q_n = (V, E)$ is Hamiltonian for every $n \geq 2$. We prove the claim by induction. Since Q_2 is the cycle with 4 vertices, the case $n = 2$ holds. Let $n \geq 3$. Suppose Q_{n-1} is Hamiltonian. We partition the set $V(Q_n) = V_1 \cup V_2$ into two disjoint sets $V_0 := \{(u_1, \dots, u_n) : u_i \in \{0, 1\}, u_n = 0\}$ and $V_1 := \{(u_1, \dots, u_n) : u_i \in \{0, 1\}, u_n = 1\}$. We note that the induced graphs $G_1 = (V_1, E \cap \binom{V_1}{2})$ and $G_2(V_2, E \cap \binom{V_2}{2})$ are isomorphic to Q_{n-1} . By the induction hypothesis, G_1 has a Hamiltonian cycle

$$H_1 = x_1 x_2 \dots x_{m-1} x_m x_1,$$

and G_2 has a Hamiltonian cycle

$$H_2 = y_1 \dots y_{m-1} y_m y_1,$$

with $x_i = (v_1, \dots, v_{n-1}, 0)$ and $y_i = (v_1, \dots, v_{n-1}, 1)$ for all $i = 1, \dots, m$. (Here $m = 2^{n-1}$, but we do not need that.). In particular, $x_i y_i \in E$ for all $i = 1, \dots, m$. Therefore,

$$x_1 x_2 \dots x_{m-1} x_m y_m y_{m-1} \dots y_1 x_1$$

is a Hamiltonian cycle of Q_n .

Exercise 2

[Optimality of Dirac's theorem]

1. **Variant 1:** We notice that a graph with an articulation point does not have a Hamiltonian cycle. Figure 10 shows the only such graph that satisfies the given conditions.

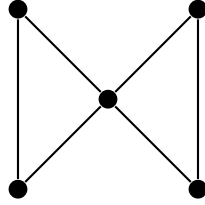


Abbildung 10: Exercise 1(ii), Var. 1

Variant 2: We observe that $6 \leq |E| \leq 10$ ($|E| = 5$ gives $G = C_5$; cycles are Hamiltonian, $|E| < 5$ and minimum degree 2 does not work).

We consider $|E| = 6$, then $\sum_{i=1}^5 d(i) = 12$. So the following cases are possible:

- a) There is a degree 4 node and 4 of degree 2,
- b) There are 2 degree 3 nodes and 3 of degree 2.

a) leads to the graph from Variant 1.

In case b) we consider that the degree 3 nodes cannot be adjacent (otherwise there is a Hamilton cycle). Thus, the graph remains as shown in Figure 11. It is easy to see that this does not contain a Hamilton cycle.

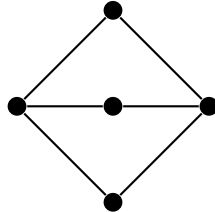


Abbildung 11: Exercise 1(ii), Var. 2

2. Let G be the complete K_{n-1} graph, to which an additional vertex v is added, connected to exactly one vertex of K_{n-1} . Then G has exactly $\binom{n-1}{2} + 1$ edges. Since v does not lie on any cycle, G does not have a Hamiltonian cycle.
3. Let G_1 be a complete graph on $\lceil \frac{n}{2} \rceil$ vertices, and let G_2 be a complete graph on $\lfloor \frac{n}{2} \rfloor$ vertices, disjoint from G_1 . Let $u \in V(G_1)$ be a vertex connected to all vertices in G_2 . The resulting graph has minimum degree $\lceil \frac{n}{2} \rceil - 1$ and is non-Hamiltonian, as at least 4 edges incident to u would need to be used.

Exercise 3

[Ore's theorem]

1. The cycle of length $n = 5$ is Hamiltonian, but for any two non-adjacent vertices u and v , we have $d(u) + d(v) = 4 < n$. Hence the reciprocal is false.
2. The contrapositive of Ore's theorem is : *If G is not Hamiltonian, then there exists two non-adjacent vertices u and v such that $d(u) + d(v) \leq n - 1$.*
3. (a) As H is maximally non-Hamiltonian, add any edge will create an Hamiltonian cycle, so adding the edge uw creates an Hamiltonian cycle $(v_1, v_2, \dots, v_n, v_1)$ with $v_1 = u$ and $v_n = w$. Removing back the edge uw yields an Hamiltonian path in $H : (v_1, v_2, \dots, v_n)$ with $v_1 = u$ and $v_n = w$.
- (b) Suppose both uv_{i+1} and wv_i are edges of H . A quick drawing (please, do the drawing!) show that $(u, v_2, \dots, v_i, w, v_{n-1}, \dots, v_{i+1}, u)$ is an Hamiltonian cycle in H . As we know H is not Hamiltonian, one of the edges uv_{i+1} or wv_i does not exist.
- (c) According to the previous question: (the total number of edges of the form uv_i for $i \in 3, \dots, n - 1$) + (the total number of edges of the form wv_{i+1} for $i \in 2, \dots, n - 2$) is at most $\#\{2, \dots, n - 2\} = n - 3$. Taking into account the edges uv_1 and wv_{n-1} (and remembering there is no edge between u and w), we get $d(u) + d(w) \leq n - 1$.
4. As we have added edges in G to obtain H , the degree $d_G(v)$ of a vertex v of G is smaller than its degree $d_H(v)$ in H , i.e. $d_G(v) \leq d_H(v)$. According to the previous question, there exists two non-adjacent vertices u and w in H (which implies there are also non-adjacent in G) such that $d_H(u) + d_H(w) \leq n - 1$. Consequently $d_G(u) + d_G(w) \leq d_H(u) + d_H(w) \leq n - 1$. This proves that Ore's theorem holds, as this is its contrapositive.

Exercise 4

[Chinese postman problem]

1. In a US city, all vertices have degree 4 (because exactly 4 streets meet at a given crossroad, as they meet at 90° angle). Thus, all vertices of the graph are of even degree, which implies there exists a Eulerian cycle (see the lecture): following this Eulerian cycle, the postman can deliver mails in every streets without passing twice in the same street.
2. According to the theorem from the lecture, there exists a Eulerian path in Osnabrück city that starts at u and ends at v . Moreover, by definition, there exists a path \mathcal{P} of length d from u to v . Following the Eulerian path, and then the path \mathcal{P} , the postman will visit every streets and then come back to its starting point by visiting a second time exactly d streets. So the minimum number of streets he needs to visit twice is at most d .
3. (a) In the trail that the postman follows, each time the postman enters a vertex w , he gets out of this vertex, so the number of time the post passes in a street adjacent to the vertex w is even. This number is $d(w) + d'(w)$, the sum of the degree of w in G and its degree in G' . Hence, except u and v , the degree of all vertices of G' is even (because it is also even in G).
- (b) According to the degree formula: $2|E| = \sum_{v \in V} d(v)$ for a connected graph $G = (V, E)$. Consequently, the number of vertices of odd degree in a connected graph (or in any connected component of a graph) is even. As u and v are the only odd vertices of G' , they can not be in different connected components (otherwise each of their connected component would have exactly 1 vertex of odd degree, and 1 is not even!).
- (c) As u and v are in the same connected component of G' , there is a path linking them. The length of this path is $\geq d$ because it is also a path in G (so especially it is longer than the *shortest* path between u and v in G). Hence the number of edges of G' is greater than d , i.e. $|E'| \geq d$. Finally, the postman can deliver all the mails by using exactly d edges twice, and whatever he does, he can not use less.

Note that if your favorite city has more than 2 vertices of odd degree, then to solve the Chinese postman problem, one should go through all matchings between vertices of odd degree, and for each compute how many edges needs to be repeated if the postman walks between the match vertices.

Exercise 5

[Bonus – Playing chess]

These questions are about finding an Hamiltonian cycle on certain graph. The graph associated to a piece of chess has for vertices the squares of the chessboard, and edges between square the piece can go to without passing over another square.

Let's do the easiest first: the Bishop can not make it because of the corners (which are isthmuses of the graph of the squares he can attain).

The Rook, the Queen and the King can make it because their graph contains a grid., see Figure 12.

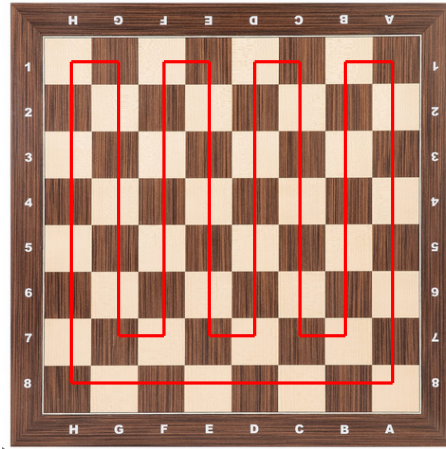


Abbildung 12: Hamiltonian cycle of Rooks, Queens and Kings.

The Knight can not make it on a 5×5 chessboard. Indeed, at each movement, the color of the square on which the Knight is changes. Consequently, if the Knight starts on a white square, then it will be after an odd number of moves is black. There are 25 squares to go through, so 25 moves to do: a Knight can not finish on the same square that it has started on, because its starting and ending squares are of different colors. The Knight can make it on a 8×8 chessboard, however, and on a lot of grids, see https://en.wikipedia.org/wiki/Knight%27s_tour.

Exercise 6

[With ChatGPT]

Remove the edge 3–4, and you get a graph with less edges, still no Hamiltonian cycle (because the vertex 6 is of degree 1), and $\deg(2) + \deg(5) = 6$. This new graph has 6 edges, which is indeed the minimum: as you need $\deg(u) + \deg(v) \geq n$ and u and v are not adjacent, then there are at least $\deg(u) + \deg(v)$ edges (thus at least 6 in our case).

Exercise 7

[Who is Hamiltonian?]

	Petersen	Herschel	Pappus	Dürer
Hamiltonian	✗	✗	✓	✓
path Hamiltonian	✓	✓	✓	✓
Eulerian	✗	✗	✗	✗
path Eulerian	✗	✗	✗	✗

For (path) Eulerian: each graph as more than 3 vertices of odd degree.

For Hamiltonian, look at the Wikipedia page of each graph.

Exercise 8

[Eulerian vs Hamiltonian]

1. Take two triangles glued at a vertex (5 vertices, 6 edges): it is Eulerian but not Hamiltonian.

2. You have 2 examples in the previous exercise.
3. Take three triangles K_3 on $\{1, 2, 3\}$, on $\{a, b, c\}$ and on $\{x, y, z\}$, and add two edges $1-a$ and $a-x$ (9 vertices, 11 edges): it is path Eulerian but not path Hamiltonian.
4. You have 4 examples in the previous exercise.
5. Impossible, see the lecture notes.
6. Take $G = P_n$ a path on n vertices: it has $n - 1$ bridges, and yet it has an Hamiltonian path (itself).

Exercise 9

[Line graph]

1. Take $G = P_3$ the path on 3 vertices. It has 2 edges, and $L(P_3)$ has 2 vertices and 1 edge, thus $L(L(P_3))$ has 1 vertex and no edge. Consequently $P_3 \neq L(L(P_3))$. For $G = K_3$, we have $L(K_3) = K_3$.
2. By definition, an Eulerian cycle in G is a cycle that goes through every edge of G once (and go back at the first edge it began with), such that consecutive edges share a vertex. Hence, following an Eulerian cycle correspond to following a cycle in the line graph $L(G)$ that goes through every edges of G , i.e. every vertices of $L(G)$ once.
3. e_{uv} is adjacent exactly to the edges of the form e_{uw} for w a neighbor of u in G (with $w \neq v$), and to e_{vw} for w a neighbor of v in G (with $w \neq u$). Thus $d_{L(G)}(e_{uv}) = (d_G(u) - 1) + (d_G(v) - 1) = d_G(u) + d_G(v) - 2$.
4. If G is Eulerian, then $d_G(u)$ is even for every vertex u of G . Hence, $d_{L(G)}(e_{uv}) = d_G(u) + d_G(v) - 2$ is even for all $uv \in E$. Thus, $L(G)$ is Eulerian as all its vertices have even degree.
5. $K_{1,3}$, the star with 1 vertex at the center connected to other 3 vertices, is not Eulerian nor Hamiltonian. $L(K_{1,3}) = K_3$ is the complete graph on 3 vertices: it is both Eulerian and Hamiltonian.

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 5

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1 [2-Connectivity]

Let $G = (V, E)$ be a minimally 2-connected graph, i.e., G is 2-connected and for every edge $e \in E$, $G \setminus e = (V, E \setminus \{e\})$ is not 2-connected.

1. Show that, in a 2-connected graph, every edge is contained in a cycle.
2. Show that a minimally 2-connected graph G has a vertex of degree 2.
3. Show that every 2-connected sub-graph H of a minimally 2-connected graph G is minimally 2-connected.

Exercise 2 [Edge and vertex connectivities]

Reminder: A graph $G = (V, E)$ is said to be ℓ -edge-connected if

- $|V| > 1$,
- $G \setminus F$ is connected for all $F \subseteq E$ with $|F| < \ell$.

Denote $\lambda(G) = \max\{\ell : G \text{ is } \ell\text{-edge-connected}\}$ the *edge connectivity* of G .

Furthermore, denote $\kappa(G)$ the *(vertex) connectivity*.

1. Show that $\kappa(G) \leq \lambda(G)$. (Hint: One possible approach is to use induction on $\lambda(G)$.)
2. Find a graph with $\kappa(G) < \lambda(G) < \Delta(G)$. Recall: $\Delta(G)$ is the maximum of the degrees of the vertices of G .

Exercise 3 [Graph toughness]

For a given real number t , a graph G is t -tough if, for every integer $k > 1$, G cannot be split into k different connected components by the removal of strictly less than tk vertices. The *toughness* $\tau(G)$ of a graph G is the maximum (or supremum, more generally) t such that G is t -tough.

1. What is the toughness of the path P_n ?
2. What is the toughness of the cycle C_n ?
3. Show that if G is not the complete graph, then $\tau(G) \leq \Delta(G)$.
4. Show that if a spanning sub-graph H of G is t -tough, then G is t -tough. Deduce that $\tau(G) \geq \max_H \tau(H)$ where the maximum is on all spanning sub-graph H of G . Find an example where $\tau(G) \neq \max_H \tau(H)$.
5. Show that if G is Hamiltonian, then G is 1-tough.
6. Show that if G is t -tough, then G is $2t$ -vertex-connected.

Exercise 4

[Blocks and cut-points]

A *cut-point* in a connected graph G is a vertex v whose deletion disconnects G . A *block* B in a connected graph G is a maximal induced sub-graph of G (with at least 2 vertices) with no cut-point (B can contain a cut-point of G). Recall: an *induced* sub-graph of a graph $G = (V, E)$, is a graph on vertex set $X \subseteq V$ with edges all the edges between the vertices of G that lie in X .

1. Show that a block is either 2-connected or it is an edge.
2. Let B_1, \dots, B_m be the blocks of G , with E_1, \dots, E_m their edges. Show that $E = E_1 \cup \dots \cup E_m$. Recall: $A \cup B$ is the disjoint union, that is $A \cup B$ with $A \cap B = \emptyset$.
3. If G is connected, show that v is a cut-point of G if and only if v belongs to (at least) two blocks of G .
4. For a graph G , with cut-points c_1, \dots, c_r and blocks B_1, \dots, B_m , the *block-cut-point graph* of G is the graph $BC(G)$ with vertex set $\{c_1, \dots, c_r\} \cup \{B_1, \dots, B_m\}$ and edges $c_i B_j$ if and only if $c_i \in B_j$. Show that if G is connected, then $BC(G)$ is a tree.

Exercise 5

[Bonus – Menger theorem on an infinite graph]

During a rainy afternoon, looking for some surprising exercise on Menger theorem, you open a math text book, and you see Figure 13.

In $\Gamma_\infty = (V, E)$, show that for all **finite** $X \subsetneq V$, there exists a path from $a \in A$ to $b \in B$ with no vertex in X . Show that it is impossible to find infinitely many **disjoint** paths that start in A and end in B . Comment this fact in regards of Menger theorem.

Submission: 28.05.2025 before 23:59

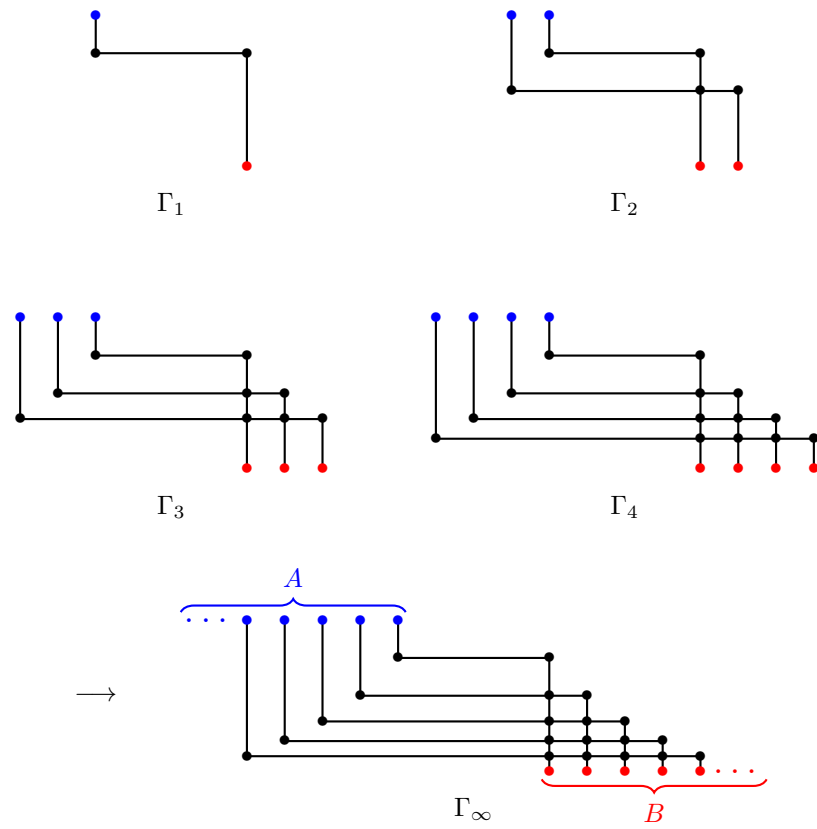


Abbildung 13: A surprising figure from an imaginary math text book.

Tutorium additional exercises

Exercise 6

[With ChatGPT]

ChatGPT told me this (it is a real copy-paste from the free version of ChatGPT):

“ Let $G_{n,n}$ be the $n \times n$ grid graph. It has n^2 vertices and is 2-connected for all $n \geq 3$. We are interested in counting how many unordered pairs of vertices $\{u, v\}$ exist such that removing both u and v disconnects the graph.

Key observations:

- Since $G_{n,n}$ is 2-connected, removing a single vertex never disconnects the graph.
- However, certain *pairs* of vertices do disconnect the graph. Such pairs are called *2-vertex cutsets*.
- The total number of unordered pairs of vertices is $\binom{n^2}{2}$, but only a small fraction of them are 2-vertex cutsets.

Conclusion: There is no known simple formula for the number of 2-vertex cutsets in $G_{n,n}$. However, for $n \geq 3$, such pairs always exist, and their number grows with n , likely in the order of $\Theta(n^2)$. This remains a challenging enumeration problem in graph theory and can be computed explicitly for small n via exhaustive search. ”

This is false. Why?

Exercise 7

[Vertex connectivity and edge connectivity]

Determine the vertex-connectivity and the edge-connectivity of:

1. the path P_n
2. the cycle C_n
3. the complete bipartite graph $K_{n,m}$ with $n \leq m$
4. the hypercube Q_n with $n \geq 1$

Exercise 8

[Some properties implying being connected]

1. Show that if a graph on n vertices has at least $\frac{(n-1)(n-2)}{2} + 1$, then it is connected.
2. Let $G = (V, E)$ be a graph with exactly two vertices of odd degree u and v . Show that $G' = (V, E \cup \{uv\})$ is connected if and only if G is connected.
3. Let $d_1 \leq d_2 \leq \dots \leq d_n$ be the degree sequence of G . Show that if $d_k \geq k$ and $k \leq n - d_n - 1$ for all k , then G is connected.
4. For a fixed n , find the smallest k such that the following holds: “For all graph G on n vertices, if G is m -regular with $m \geq k$, then G is connected.”.
5. Show that the complement of a disconnected graph is connected. Show that the reciprocal is false. the *complement* of a graph G is the graph \bar{G} on the same vertex set, where uv is an edge of \bar{G} if and only if uv is not an edge of G .

Exercise 9 [Edge connectivity split]
 Let $G = (V, E)$ be a graph with edge-connectivity $\lambda(G) = k \geq 1$. Show that there exists $X, Y \subseteq V$ such that $X \cup Y = V$ and there are exactly k edges with one endpoint in X and one endpoint in Y .

Exercise 10 [Connectivity and cycles]

1. Show that if G is k -connected with at least $2k$ vertices, then G contains a cycle of length at least $2k$.
2. Show that if G is k -connected, then any k vertices of G lie on a cycle (this cycle is not necessarily of length k).

Exercise 11 [3-connectivity]
 Show that every edge of a 3-connected graph lies on a non-separating induced cycle. A cycle C of a graph G is *induced* if there is no edge in G between its vertices except from the edge of C . A cycle is *non-separating* if $G \setminus C$ is connected.

Exercise 12 [Deletion and contraction]
 If G is k -connected, and uv is an edge of G , show that G/uv is k -connected if and only if $G \setminus uv$ is $(k - 1)$ -connected. For $G = (V, E)$, recall that $G \setminus uv = (V, E \setminus \{uv\})$, and G/uv is the graph obtained by merging the vertices u and v of G into a single vertex (which is adjacent to both the neighbors of u and the neighbors of v).

Discrete Mathematics – Solutions Sheet 5

Exercise 1

[2-Connectivity]

1. Direct from the definition in the lecture notes.
2. By Theorem 7.3, we obtain G from a cycle by adding paths between two existing vertices. If G is a cycle, then every vertex on it has even degree. Otherwise, consider the last path added in the construction of G . If this path has length at least 2, then there exists a vertex of degree 2. If not, then the last addition was an edge, and its removal results in a 2-connected graph. However, this contradicts the assumption that G was minimally 2-connected.
3. Let H be a 2-connected subgraph of G . Suppose H is not minimal. Then there exists an edge $uv \in E(H)$ such that $H \setminus \{uv\} = (V(H), E(H) \setminus \{uv\})$ is 2-connected. However, by the assumption about G , there also exists a vertex w such that $G' = G \setminus \{uv\} \setminus \{w\}$ is not connected. On the other hand, $H' = H \setminus \{uv\} \setminus \{w\}$ is 2-connected, implying that H' must be contained in a connected component C of G' . Thus, H is a subgraph of $G_{V(C) \cup \{w\}}$, and in particular, $u, v \in V(C) \cup \{w\}$. However, this implies that $G \setminus \{w\}$ is not connected.

Exercise 2

[Edge and vertex connectivities]

- If $\lambda(G) = 0$, then G is not connected. It follows that $\kappa(G) = 0$.
- If $\lambda(G) = 1$, then there exists an edge uv such that $G \setminus e$ is not connected. If $|V| = 2$, then $G \cong K_2$ and $\kappa(K_2) = \lambda(K_2) = 1$. Otherwise, $G \setminus u$ (and $G \setminus w$) is not connected, and we have $\kappa(G) = 1$.
- If $\lambda(G) = |V| - 1$, then the minimum degree of G is also $|V| - 1$ (Proof: Suppose there exists v with $d(v) < |V| - 1$. Then $G \setminus N$ is not connected, where $N = e \in E : v \in e$). A graph $G = (V, E)$ with minimum degree $|V| - 1$ is isomorphic to $K_{|V|}$, and $\kappa(K_{|V|}) = |V| - 1$.
- Suppose $1 < \lambda = \lambda(G) < |V| - 1$ and $G \setminus \{e_1 \dots e_\lambda\}$ is not connected. Let $G_1 = G \setminus e_\lambda$. We have $\lambda(G_1) = \lambda - 1$. By the induction hypothesis, $\kappa(G_1) \leq \lambda(G_1)$. Therefore, there exist vertices v_1, \dots, v_j with $j \leq \lambda - 1$, such that $G_2 = G_1 \setminus \{v_1 \dots v_j\}$ is not connected. Since $|V| > \lambda + 1 > 2$, G has at least three vertices. Let $e_k = uv$.
 - If $u, v \in V(G_2)$ and $G_2 \cup e_k$ is connected, then $(G_2 \cup e_k) \setminus u = G \setminus \{v_1 \dots v_j, u\}$ is not connected. It follows that $\kappa(G) \leq j + 1 \leq \lambda$.
 - If $u, v \in V(G_2)$ and $G_2 \cup e_k$ is not connected, then $G \setminus \{v_1 \dots v_j\}$ is not connected. It follows that $\kappa(G) \leq j \leq \lambda - 1 < \lambda$.

- If $u \notin V(G_2)$ (or $v \notin V(G_2)$), then $G_2 = G \setminus \{v_1 \dots v_j\}$. Since G_2 is not connected, it follows that $\kappa(G) \leq j \leq \lambda - 1 < \lambda$.

Exercise 3

[Graph toughness]

1. Removing a vertex that is not an endpoint in the path P_n split it into 2 connected components which are themselves paths (and it creates 2 new endpoints). In order to split the path P_n into k connected components, one need to remove at least $k - 1$ vertices (assuming that $n > 2k$). Hence, P_n cannot be split into k different connected component by the removal of $< k - 1$ vertices: if P_n is t -tough, then $tk \leq k - 1$ for all $k \geq 2$. Thus $\tau(P_n) \leq \frac{k-1}{k}$ for all $k \geq 2$, and we deduce that $\tau(P_n) \leq \frac{1}{2}$. We also see that P_n is $\frac{1}{2}$ -tough. Hence the toughness of the path is $\tau(P_n) = \frac{1}{2}$.
2. Removing any vertex of the cycle C_n maintain 1 connected component, which is a path P_n . Hence, to split C_n into k connected components, one needs to remove k vertices (or more). Hence the toughness of the cycle is $\tau(C_n) = \max\{t ; \forall k \geq 2, t \leq \frac{k}{k}\} = 1$.
3. Suppose $uv \notin E$, then the removal of all the neighbors of u and all the neighbors of v (which is less than $2\Delta(G)$ vertices) will create $\kappa \geq 2$ connected components (the one of u , the one of v , and maybe others). Consequently, if t is such that G can not be split in κ connected component by the removal of $< t\kappa$ vertices, then $t\kappa \leq 2\Delta(G)$; which means $\tau(G) \leq \Delta(G)$ (recall $\kappa \geq 2$).
4. If H is a spanning sub-graph of G , and $X \subseteq V$, then note that $H \setminus X$ has more connected components than $G \setminus X$ (because G contains all the edges of H). If H is moreover t -tough, then for any $X \subseteq V$ with $|X| < tk$, the number of connected components of $H \setminus X$ is $< k$, and hence, the number of connected components of $G \setminus X$ is $< k$. That means G is t -tough. As $\tau(G)$ is the supremum of the t such that G is t -tough, we immediately get $\tau(G) \geq \tau(H)$ for all spanning sub-graph H of G , and the maximum follows. The complete graph satisfies $\tau(K_n) = +\infty$ (because K_n cannot be split in k connected components by the removal of tk vertices, for all $t > 0$), and any spanning sub-graph has finite toughness, so $\tau(G) \neq \max_{H \text{ spanning}} \tau(H)$.
5. If G is Hamiltonian, then it has an Hamiltonian cycle, so C_n is a spanning sub-graph of G . As C_n is 1-tough, G is 1-tough.
6. If G is t -tough, then, taking $k = 2$, we get that G cannot be split in 2 connected components (i.e. disconnected) by the removal of $2t$ vertices. Hence, G is $2t$ -vertex-connected, by definition.

Exercise 4

[Blocks and cut-points]

1. Edges and 2-connected graphs do not contain a cut-point. Conversely, if G does not contain a cut-point, then let u and v be vertices of a block B of G . If B is not an edge, then, as it has no cut-points, it can only be disconnected by the removal of ≥ 2 vertices: B is 2-connected. For what follows, we will take the convention that an edge is 2-connected.
2. Fix $uv \in E$, and consider the biggest $X \subseteq V$ such that $u \in X$, $v \in X$, and $B := G|_X$ (the sub-graph of G induced on X) is 2-connected. On the one hand, uv is an edge of B . On the other hand, B contains no cut-point (because it is 2-connected), and any Y with $X \subsetneq Y$ will give rise to $G|_Y$ containing a cut-point (by maximality of X). Hence, B is a block of G . We have proven that $E = E_1 \cup \dots \cup E_m$. To see that $E_i \cap E_j = \emptyset$, just note that if B_i and B_j intersect on (at least) an edge uv , then $B_i \cup B_j$ can not be disconnected by removing 1 vertex (because there is no cut-point in $B_i \setminus \{u, v\}$ nor in $B_j \setminus \{u, v\}$, and the removal of u or of v does not disconnect $B_i \cup B_j$, as the other vertex ensures connectivity). This implies that every edge sits in a unique, well-defined block.
3. If v belongs to two blocks B and B' , then if v is not a cut-point of G , that implies $B \cup B'$ has no cut-point, and hence is 2-connected: this would contradict the maximality of B and of B' . Thus v is a cut-point. Conversely, if v is a cut-point of G , suppose all its neighbors are in the same block B . Then $B \setminus v$ is connected, hence all the neighbors of v can be pairwise connected by paths that avoid v . The removal of v in G does not disconnect the graph. This contradiction shows that v belongs to at least two blocks.
4. If G is connected, then $BC(G)$ is connected because the construction of $BC(G)$ amounts to replacing each block by a vertex connected to all the cut-points this block contains, which does not infringe the connectivity of G . Take a connected sub-graph of $BC(G)$, encapsulating the blocks B_1, \dots, B_r , and let $X = \bigcup_i B_i$. Then $G|_X$ is connected, indeed each $G|_{B_i}$ is connected, and if $B_i c$ and $B_k c$ are edges of $BC(G)$, then $G|_{B_i \cup B_j}$ is connected (because $G|_{B_i}$ and $G|_{B_j}$ intersect on c). Suppose $BC(G)$ has a cycle, and let B be the union of the blocks appearing in this cycle, and let $v \in B$. We prove that $B \setminus v$ is connected. If v is not a cut-point, then v is inside a block and does not disconnect the cut-points of this block, hence $B \setminus v$ is connected. If v is a cut-point, then the block-cut-point graph of $B \setminus v$ is obtained from $BC(B)$ by removing v , hence it is connected, and so is $B \setminus v$. Consequently, $BC(G)$ is a tree (a connected graph without cycle).

Exercise 5

[Bonus – No Menger theorem for infinite subsets]

The graph Γ_∞ has infinitely many vertices and infinitely many edges. However, note that each vertex belongs to finitely many edges. Let enumerate the vertices in A from right to left a_1, a_2, a_3, \dots , and the vertices of B from left to right b_1, b_2, b_3, \dots . For each n , consider the path P_n from a_n to b_n defined by following

3 line segments, the vertical one starting at a_n , going down vertically (to the only neighbor of a_n), then going horizontally right until being at the vertical of b_n , and then going down to b_n . Each vertex of Γ_∞ belongs to finitely many paths P_n . Hence, for any **finite** $X \subsetneq V$, there is a path P_n that avoids all vertices in X : no finite vertex subset X separate A from B .

On the other hand, consider a collection of disjoint paths $(Q_j)_j$ from A to B . Note that Q_j is not necessarily of the form P_n described above. Let b be the leftmost endpoint of some Q_j in B , and $a \in A$ the other endpoint of Q_j . Suppose there is a path Q_i whose endpoint in A is on the left of a . By construction of b , the endpoint of Q_i in B is on the right of b : that means Q_i intersects Q_j , so they are not disjoint. Hence, a is the leftmost endpoint present among all Q_j , which implies there are finitely many different Q_j (at most k such that $a = a_k$).

Menger theorem ensure that, in a finite graph G , the number of disjoint paths between two vertices u and v is the same as the number of vertices that one needs to remove in order to separate u from v . This also works if one takes subsets of the vertices : for $A, B \subseteq V$, the number of disjoint paths with one endpoint in A and one in B is the same as the number of points needed to remove in order to separate A from B . We have just seen that, in infinite graphs, this property does not hold, as Γ_∞ is a counter-example.

Exercise 6

[With ChatGPT]

Due to the properties of the grid graph, the only way to disconnect $G_{n,n}$ by removing exactly two vertices is to take pick a corner v of $G_{n,n}$ (i.e. a vertex with exactly 2 neighbors), and to remove its 2 neighbors. As $G_{n,n}$ has 4 corner, there are precisely 4 ways to disconnect $G_{n,n}$ by removing two vertices. ChatGPT told there are $\Theta(n)$: this is utter bullshit.

Discrete Mathematics – Summer Semester 2025

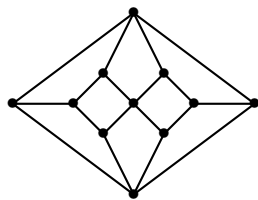
Exercise Sheet 6

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

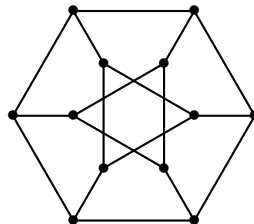
Exercise 1

[Kuratowski's Theorem]

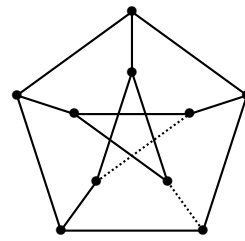
1. Which of Herschel, Dürer, and Petersen graphs are planar, and why? (Hint: consider removing the dotted edges in the Petersen graph.)
2. Decide for which $m, n > 0$ the following graphs are planar:
 - (a) complete graph K_n
 - (b) complete bipartite graph $K_{m,n}$
 - (c) hypercube graph Q_n
3. Let $G = (V, E)$ be a minimal non-planar graph: Does there exist an edge $e \in E$ such that $G \setminus e$ is maximal planar? *Minimal non-planar*: for all $e \in E$, $G \setminus e$ is planar. *Maximal planar*: for all $e \notin E$, adding e to G makes it not planar.



Herschel graph



Dürer graph



Petersen graph

Exercise 2

[Euler's Formula]

Let G be a connected planar graph with minimum degree at least 3. Let v_i be the number of vertices of G with degree i . Show that if G is a triangulation, i.e., in a planar drawing, each face is bounded by 3 edges, then (with δ the maximum degree):

$$3v_3 + 2v_4 + v_5 = v_7 + 2v_8 + 3v_9 + \cdots + (\delta - 6)v_\delta + 12$$

Exercise 3

[Triangulations]

Recall that a *triangulation* is a connected planar graph such that all regions are bounded by 3 edges.

1. Let $T = (V, E)$ be a triangulation. Show that $|V|$ is even if and only if $|E|$ is even.
2. How many vertices has a triangulation with 42 edges (justify it)?

Exercise 4

[Planar Drawings]

Let G be a connected planar graph with n nodes, whose regions are bounded by at most 4 edges. Furthermore, there is no vertex of degree 1. Show that:

1. Any planar drawing of G has at least $n - 2$, and at most $2n - 4$ regions.
2. When does every planar drawing of G have exactly $n - 2$, and when exactly $2n - 4$ regions?

Exercise 5

[Bonus – Sharing a pizza]

You invite some friends at your place, and it is now time to share the pizza. Your friends are very conciliatory, so they would be happy with any part of the pizza (no matter the size, shape, nor if it contains crust). With your knife, the only possibility is to cut the pizza by making a straight line from a point of the crust to another point of the crust.

How many cuts do you need to satisfy you and your friends, i.e. 220 people?

Submission: 04.06.2025 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

ChatGPT told me this (it is a real copy-paste from the free version of ChatGPT):

“ **Maximum Number of Edges in a Planar Bipartite Graph** Let G be a bipartite planar graph with $n \geq 3$ vertices. Then:

$$|E(G)| \leq 2n - 4$$

Justification: Since G is bipartite, all faces in a planar embedding must have length at least 4 (no odd cycles). Using Euler's formula $n - m + f = 2$ and the inequality $4f \leq 2m$, we derive:

$$n - m + \frac{m}{2} \geq 2 \quad \Rightarrow \quad m \leq 2n - 4$$

Example: The planar grid graph $P_k \times P_\ell$ has $n = k\ell$ vertices and $2n - 2k - 2\ell + 2$ edges. For large k, ℓ , this approaches the bound $2n - 4$. ”

This is false. Why (careful: this one is inconspicuous)?

Exercise 7

[Bound on degree of planar graphs]

1. Prove there is no planar graph with 6 vertices with at least 3 vertices of degree 5.
2. Find the maximum number a of edges a planar graph on 6 vertices can have, and draw a planar graph on 6 vertices with a edges.

Exercise 8

[(Football) ball]

A ball (made for a football game), is a polyhedron made of hexagons of pentagons. We admit that the graph of a 3-dimensional polyhedron is always planar (the graph G of a polyhedron is the graph whose vertices are the vertices of the polyhedron and whose edges are the edges of the polyhedron). Find the number of vertices, edges, pentagons and hexagons a ball can have.

Exercise 9

[Platonic solids]

A *Platonic solid* is a 3-dimensional (convex) polyhedron whose faces are all regular polygons with the same number of sides, and whose vertices all have the same number of adjacent edges. We admit that the graph of a polyhedron is a planar graph (the graph G of a polyhedron is the graph whose vertices are the vertices of the polyhedron and whose edges are the edges of the polyhedron). Show there are only 5 Platonic solids. Who are they?

Exercise 10

[Tutte graph]

In 1884, Peter Tait (whose Wikipedia's photograph is amazing), conjectured that: “Every 3-connected planar cubic graph has a Hamiltonian cycle (along the edges) through all its vertices”. Recall that *cubic* means that each vertex has degree 3. Look at the graph produced by William Tutte in 1946, in Figure 14. What do you think?

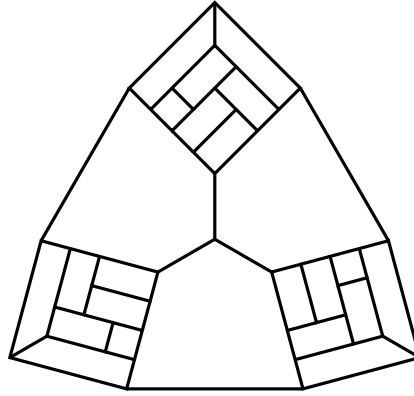


Abbildung 14: Tutte's graph

Discrete Mathematics – Solutions Sheet 6

Exercise 1

[Kuratowski's theorem]

1. Herschel graph is clearly planar (from its drawing). Dürer graph is planar, see Figure 15 (Left), only 3 vertices have been moved. Petersen graph is not planar as it contains a subdivision of $K_{3,3}$, see Figure 15 (Right).
2. (a) For $n \leq 4$, K_n is planar. For $n \geq 5$, K_n contains a K_5 subgraph and thus, by Kuratowski's theorem, is not planar.
 - (a) If $n \leq 2$, then $K_{n,m}$ can be drawn planarly as shown in Figure 16. Analogously, $K_{n,m}$ is planar for $m \leq 2$. If $m, n \geq 3$, then $K_{n,m}$ contains a $K_{3,3}$ subgraph and is thus not planar.
 - (a) For $n \leq 3$, Q_n is planar. In Q_3 , only one path is missing for a $K_{3,3}$ subdivision (see Figure 17a). This path exists in Q_4 . Since $Q_n \subseteq Q_{n+1}$ for every n , it follows that Q_n is not planar for $n \geq 4$.
3. No, because: We know from the lecture that $K_{3,3}$ is minimally non-planar. If we remove an edge from $K_{3,3}$, we can add another edge and obtain a planar graph (see Figure 17b, the dotted edge is the newly added one).

Exercise 2

[Euler's formula]

Since G is a triangulation, each face contains 3 edges on its boundary. On the

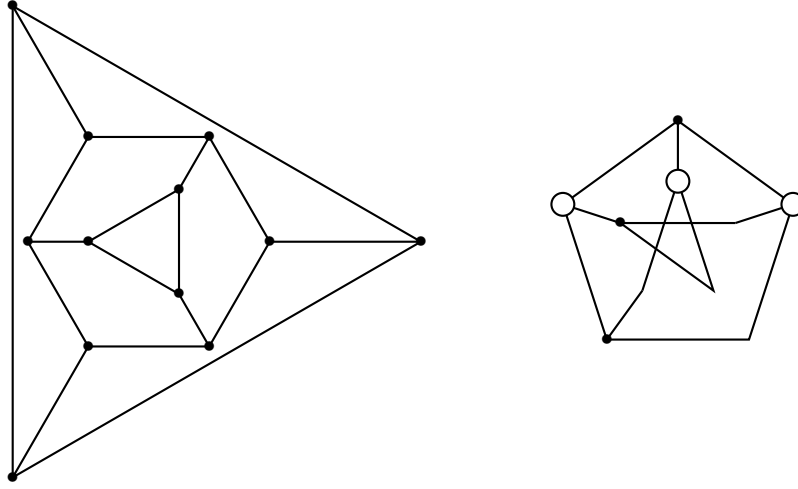


Abbildung 15: Left: Planar drawing of Dürer graph, only 3 vertices have been moved. Right: Subdivision of $K_{3,3}$ inside Pertesen graph, note that 4 vertices have been removed (and two edges deleted).

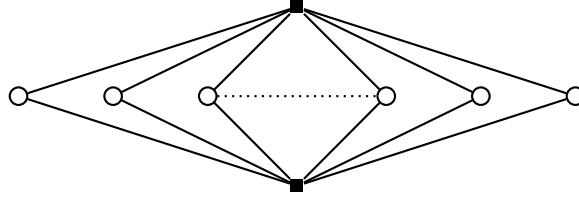


Abbildung 16

other hand, each edge lies on the boundary of exactly 2 faces. Double counting yields

$$3g = 2|E|, \quad \text{that is,} \quad g = \frac{2}{3} \cdot |E|.$$

Furthermore, we have

$$n = \sum_{i=3} v_i$$

by definition of v_i . The last relation we will use is

$$2|E| = \sum_{v \in V} d(v) = \sum_{i=3}^{\Delta} v_i \cdot i,$$

i.e., $|E| = \frac{1}{2} \cdot \sum_{i=3}^{\Delta} v_i \cdot i$. In particular, from (), we have

$$g = \frac{1}{3} \cdot \sum_{i=3}^{\Delta} v_i \cdot i.$$

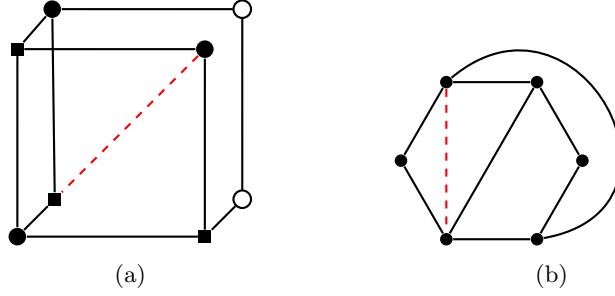


Abbildung 17

Substituting all of this into the Euler's formula yields

$$\begin{aligned}
 2 &= v_3 + \cdots + v_\Delta - \frac{1}{2} \cdot \sum_{i=3}^{\Delta} v_i \cdot i + \frac{1}{3} \cdot \sum_{i=3}^{\Delta} v_i \cdot i \\
 &= v_3 + \cdots + v_\Delta - \frac{1}{6} \cdot \sum_{i=3}^{\Delta} v_i \cdot i.
 \end{aligned}$$

Multiplying both sides by 6 gives

$$12 = \sum_{i=3}^{\Delta} (6 - i) v_i,$$

from which the claim follows after a slight rearrangement.

Exercise 3

[Triangulations]

1. Let F be the set of faces in a planar drawing. We count the elements in $M = \{(e, f) : e \in E, f \in F, e \in f\}$. Since each face is bounded by three edges, we have $|M| = 3|F|$. Since each edge is on the boundary of two faces, we have $|M| = 2|E|$. Thus, $2|E| = 3|F|$ and $|F|$ is even. Using Euler's formula, we deduce that $|V| - |E| = 2 - |F|$ is even. Hence, $|V|$ is even (odd) if and only if $|E|$ is even (odd).
2. We need to solve the following system of equations:

$$\begin{cases} |V| - |E| + |F| = 2 \\ 2|E| = 3|F|. \end{cases}$$

Thus, every triangulation with 42 edges has exactly $|V| = 2 + |E|/3 = 2 + 14 = 16$ vertices.

Exercise 4

[Planar drawings]

1. Since there are no vertices of degree 1, each edge must be on the boundary of two faces, and each face has at least 3 edges. By double counting, we have

$$\begin{aligned}
2 \cdot |E| &= \sum_{e \in E} |\{X \text{ face} : e \text{ is on the boundary of } X\}| \\
&= \sum_{X \text{ face}} |\{e \in E : e \text{ is on the boundary of } X\}| \\
&\begin{cases} \leq 4g & (\text{since each face has at most 4 edges}) \\ \geq 3g & (\text{since each face has at least 3 edges}). \end{cases}
\end{aligned}$$

Thus, $3g \leq 2|E| \leq 4g$. From Euler's formula, $|E| = n + g - 2$. Substituting this, we get

$$3g \leq 2n + 2g - 4 \leq 4g,$$

which implies $n - 2 \leq g \leq 2n - 4$.

2. Equality holds if and only if the above estimations become equations, i.e., $2|E| = 4g$ or $2|E| = 3g$, which means each face is bounded by 4 or 3 edges, respectively.

Exercise 5

[Bonus – Sharing a pizza]

Let S be the number of pieces. We consider the graph G , see Figure 18, whose nodes are the intersection points of every 2 cuts and the intersection points of the cuts with the boundary. The edges of G are the resulting subdivision of the cuts and the boundary.

To maximize the number of pieces, we make the cuts such that every two cuts intersect, and no three cuts intersect at the same point.

Since G is a planar graph, we have $S = |F(G)| - 1 = |E(G)| - |V(G)| + 1$.

- Each cut intersects the boundary twice, and any two cuts intersect. Hence, $|V(G)| = 2n + \binom{n}{2}$.
- The boundary is divided into $2n$ edges. Additionally, each cut is divided into n edges because there are exactly $n + 1$ nodes on each cut. Therefore, $|E(G)| = 2n + n \times n$.

Now we have

$$\begin{aligned}
S &= |F(G)| - 1 = |E(G)| - |V(G)| + 1 \\
&= 2n + n \cdot n - 2n - \binom{n}{2} + 1 = \frac{n^2 + n + 2}{2}.
\end{aligned}$$

Now, if I have 219 friends (so 220 people to feed, including myself), I want to find the least n such that $\frac{n^2 + n + 2}{2} \geq 220$, which is $n = 21$. This will make 232

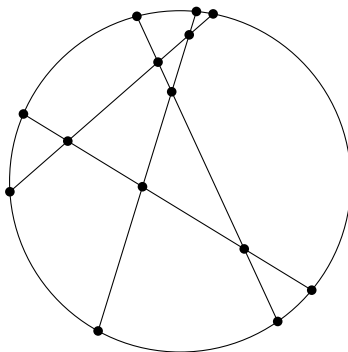


Abbildung 18: A pizza.

pieces. Good luck for making theses cuts (you will need a large pizza in order to not have too thin pieces)!

Exercise 6

[With ChatGPT]

The reasoning is correct, but the final example is false. ChatGPT does not know the number of edges of a grid graph (which should be denoted $P_k \square P_\ell$, by the way): the correct number of edges is $k\ell - k - \ell$ or $n - k - \ell$.

Exercise 7

[Bound on degree of planar graphs]

1. Let u, v, w be the vertices of degree 5, and a, b, c the other vertices. As $5 = 6 - 1$, these vertices are connected to all the other. In particular, G has a sub-graph $K_{3,3}$ because all 3 vertices u, v, w are pair-wise connected to all 3 vertices a, b, c . Kuratowski theorem ensures G is not planar.
2. If G is planar, then $|E| \leq 3|V| - 6$, which means $|E| \leq 3 \times 6 - 6 = 12$, for 6 vertices. Any triangulation on 6 vertices does the job.

Exercise 8

[(Football) ball]

We have several equations to respect. Let F be the number of faces, E the number of edges, and V the number of vertices of the ball, P the number of pentagons and H the number of hexagons.

Clearly: $F = P + H$.

Furthermore (Euler's formula): $F - E + V = 2$.

Each edge is in exactly two faces, each hexagon has 6 edges, and each pentagon has 5 edges, so: $5P + 6H = 2E$.

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 7

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1

[Graph Colorings]

1. Compute the chromatic number $\chi(G)$ for the following graphs:
 - (a) The path $G = P_n$
 - (b) The cycle $G = C_n$
 - (c) The complete graph $G = K_n$
 - (d) The complete bipartite graph $G = K_{n,m}$
2. Let $G = (V, E)$ be a graph with $\chi(G) = k$. Further, let $f : V \rightarrow [k]$ be a coloring of G . Show that for all $\{i, j\} \in \binom{[k]}{2}$, there exists an edge $uv \in E$ such that $f(u) = i$ and $f(v) = j$.
3. Conclude that:

$$\chi(G) \leq \frac{1}{2} + \sqrt{2|E| + \frac{1}{4}}$$

Exercise 2

[Chromatic polynomials]

Let $G = (V, E)$ be a graph and $e = uv \in E$. We define the graph G/e as follows: $V(G/e) := (V \setminus \{u, v\}) \cup \{\tilde{u}\}$ and $E(G/e) := \{e \in E : u, v \notin e\} \cup \{\tilde{u}w : w \neq u, w \neq v, uw \in E \text{ or } vw \in E\}$. (The graph G/e is obtained from G by contracting the edge e).

Let $P_G(k)$ denote the number of colorings of G with k colors.

1. Show that $P_G(k) = P_{G \setminus e}(k) - P_{G/e}(k)$ holds for every $k \geq 0$.
2. Determine $P_{K_n}(k)$ depending on k and n . Hint: Use 1. and an induction over n .
3. Show that the function $P_G : \mathbb{N} \rightarrow \mathbb{N}$, $k \mapsto P_G(k)$ is a polynomial function in k of degree $|V|$.

Exercise 3

[Outerplanar graphs]

A planar graph $G = (V, E)$ is called *outerplanar* if it has a planar drawing in which all nodes lie on the boundary of the unbounded region. Show that:

1. If G is outerplanar with maximum degree 2, then G is 3-colorable.

2. If G is outerplanar with a vertex u of degree at least 3, then G can be splitted into two smaller outerplanar graphs H_1 and H_2 such that $G = H_1 \cup H_2$ and $H_1 \cap H_2$ is an edge.
3. Every outerplanar graph is 3-colorable.

Exercise 4

[Edge colorings]

Let $G = (V, E)$ be a graph. Show:

1. If $c : E \rightarrow [k]$ is an edge coloring of G , then every set $E_i = \{e \in E : c(e) = i\}$ forms a matching of G . An *edge coloring* is a coloring of the edge of G such that adjacent edges get different colors.
2. If G is r -regular and $\chi'(G) = \Delta(G)$, then G has a matching that covers V .
Note that $\chi'(G)$ is the minimum number of colors needed such that an edge coloring exists.
3. If G is r -regular and $|V|$ is odd, then $\chi'(G) = \Delta(G) + 1$.

Exercise 5

[Bonus – Ramsey theory]

You draw a complete graph of size n on your paper, when a friend comes by. He proposes the following game: you color some edges in red, then he colors some edges in blue, then you color some edges in red, and so on... The looser is the first person to create a monochromatic triangle (i.e. to color three pair-wise adjacent edges with the same color). Show that it is possible that no one loses if $n = 5$, but there will always be a looser for $n \geq 6$.

The next day, you meet your little sister, and you propose the same game. She is cunning, so she asks that you loose if you create a red triangle, but she loses only if she create a blue K_4 . What is the minimal value of n you need to pick such that the game has a looser ? Hint: You can prove (or use directly) that $R(r, s) < R(r - 1, s) + R(r, s - 1)$ when $R(r - 1, s)$ and $R(r, s - 1)$ are even, where $R(r, s)$ is the minimum n such that there will be either a red K_r or a blue K_s when coloring the edges of K_n in red and blue.

Submission: 11.06.2025 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

ChatGPT told me this (it is a real copy-paste from the free version of ChatGPT):

“ **Number of 3-colorable simple graphs on 5 vertices**

There are $\binom{5}{2} = 10$ possible edges in a simple graph on 5 vertices, so the total number of simple graphs is:

$$2^{10} = 1024.$$

To count the number of 3-colorable graphs, we subtract those with chromatic number at least 4.

On 5 vertices:

1. There is exactly one graph with chromatic number 5, namely the complete graph K_5 .
2. There are 5 graphs (other than K_5) with chromatic number 4.

Thus, the number of graphs with chromatic number ≥ 4 is 6.

Therefore, the number of 3-colorable graphs is: $1024 - 6 = \boxed{1018}$. ”

This is false. Why?

Exercise 7

[Hypercube graph]

Prove that for $n \geq 1$, the hypercube graph Q_n is 2-colorable.

Exercise 8

[Petersen graph]

For G the Petersen graph, compute $\chi(G)$.

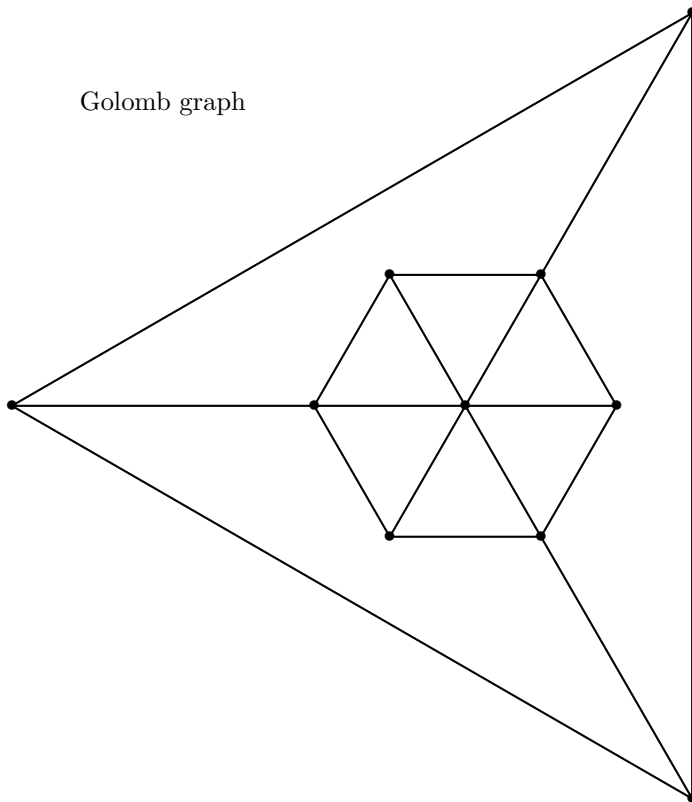
Exercise 9

[Hadwiger–Nelson problem]

We consider the following infinite graph Γ : the set of vertices of Γ are all the points of the plane \mathbb{R}^2 ; there is an edge pq if the points p and q are at (euclidian) distance exactly 1. The *Hadwiger–Nelson problem* ask for computing $\chi(\Gamma)$.

1. Tile the plane with regular hexagons of diameter $1 - \varepsilon$ with $\varepsilon > 0$ very small. Show that one can color these hexagons with 7 colors in a way that proves $\chi(\Gamma) \leq 7$.
2. Show that Golomb graph G (see below) can be drawn in the plane such that adjacent vertices are at distance 1.
3. Show that $\chi(G) = 4$, and deduce that $\chi(\Gamma) \geq 4$.
4. **N.B.** The current research [May 2024] states that $\chi(\Gamma) \in \{5, 6, 7\}$, but the actual value is unknown. If you have plenty of time, solve this problem!

Golomb graph



Exercise 10

[Greedy coloring]

Let $G = (V, E)$ be a graph with $V = \{1, \dots, n\}$. The *greedy coloring* of G is the coloring $c : V \mapsto \mathbb{N}$ with $c(j) = \min(\mathbb{N} \setminus \{c(i) ; 1 \leq i < j \text{ and } ij \in E\})$. The *greedy chromatic number* $\chi_g(G)$ is the number of colors used by the greedy number.

1. Show that the greedy coloring is a (vertex) coloring.
2. Show that two isomorphic graphs do not necessarily have the same greedy chromatic number.
3. Show that for any graph G isomorphic to the complete bipartite graph $K_{n,m}$, we have $\chi_g(G) = 2$.
4. Let $G = (V, E)$ with $V = \{1, \dots, 2n\}$ and $E = \{\{i, j\} ; 1 \leq i \leq n \text{ and } n+1 \leq j \leq 2n, j \neq i+n\}$; and let $G' = (V, E')$ with $E' = \{\{i, j\} ; i \text{ odd, } j \text{ even, } j \neq i+1\}$. Show that G and G' are isomorphic, but $\chi_g(G) = 2$ whereas $\chi_g(G') = n$.
5. For $X \subseteq \mathbb{R}$, let $\text{range}(X) = \max(X) - \min(X)$. Deduce there exists a sequence of graphs G_n such that $\text{range}(\chi_g(H) ; H \text{ isomorphic to } G_n) \rightarrow +\infty$ when $n \rightarrow +\infty$.
6. For any graph G , show that there exists a graph H , isomorphic to G such that $\chi_g(H) = \chi(G)$.

Exercise 11

[Interval graph]

Let $G = (V, E)$ be a graph with $V = \{1, \dots, n\}$. The *greedy coloring* of G is the coloring $c : V \mapsto \mathbb{N}$ with $c(j) = \min(\mathbb{N} \setminus \{c(i) ; 1 \leq i < j \text{ and } ij \in E\})$.

Let $\mathcal{I} = (I_1, \dots, I_r)$ be a collection of intervals of \mathbb{R} . The associated *interval graph* $G_{\mathcal{I}}$ is the graph on vertex set \mathcal{I} and edges between I_j and I_k if $I_j \cap I_k \neq \emptyset$.

1. For each $I_k \in \mathcal{I}$, let $x_k = \min I_k$. Compute $\chi_g(H)$ for the graph H , isomorphic to $G_{\mathcal{I}}$ obtained by ordering the intervals of \mathcal{I} according to x_k .
2. So that $\chi_g(H) = \chi(G_{\mathcal{I}})$ for the above-defined H .
3. You are responsible for organizing the schedule of the university: each professor tell you when he/she will teach (several time slots, each with departure time and finishing time), and you need to tell them in which room they will teach. Two professors cannot occupy the same room at the same moment. Design an algorithm with linear complexity to solve this problem.

Exercise 12

[Grundy number]

Let $G = (V, E)$ be a graph with $V = \{1, \dots, n\}$. The *greedy coloring* of G is the coloring $c : V \mapsto \mathbb{N}$ with $c(j) = \min(\mathbb{N} \setminus \{c(i) ; 1 \leq i < j \text{ and } ij \in E\})$. The *greedy chromatic number* $\chi_g(G)$ is the number of colors used by the greedy number. The *Grundy number* is $\Psi(G) = \max(\chi_g(H) ; H \text{ isomorphic to } G_n)$.

1. Compute $\Psi(G)$ for the path P_n .
2. Compute $\Psi(G)$ for the cycle C_n .
3. Show that if G contains a triangle (i.e. 3 pair-wise connected vertices), or 4 vertices such that the edges between them form a path, then $\Psi(G) \neq \chi(G)$.
4. Deduce that the complete bipartite graphs $K_{n,m}$ are the only connected graphs for which $\Psi(G) = 2$.
5. A graph is *well-colored* if $\Psi(G) = \chi(G)$. Show that $K_{2,2,2}$ (graph obtained from K_6 by removing the edges 14, 25 and 36) is well-colored.
6. Show that any complete multi-partite graph is well-colored.
7. Show that $\Psi(G) \geq k$ if and only if the graph obtained from G by adding a $(k-1)$ -vertex clique is well-colored.

Exercise 13

[Grundy game]

The *Grundy game* plays as follows:

“Given a graph $G = (V, E)$, two players, Alice and Bob, alternate their turns in choosing uncolored vertices to be colored. Whenever an uncolored vertex is chosen, it is colored by the least positive integer not used by any of its colored neighbors. Alice’s goal is to minimize the total number of colors used in the game, and Bob’s goal is to maximize it.”

Play the Grundy game on a graph of your choice (especially, path, tree, cycle, complete graph, complete bipartite graph).

Discrete Mathematics – Solutions Sheet 7

Exercise 1

[Graph Colorings]

1. (a) $\chi(P_n) = 2$ for $n \geq 2$, because P_n is bipartite.
 (b) $\chi(C_n) = 2$ if n is even (because then C_n is bipartite), and $\chi(C_n) = 3$ if n is odd (because removing a vertex give rise to a path, which is 2-colorable, and then we can color the last vertex with a third color).
 (c) Any two vertices in K_n are connected. Therefore, $\chi(K_n) = n$.
 (d) The vertex set of $K_{m,n}$ is the disjoint union of two sets A and B , where no vertices are connected within each set. We can color the vertices in A with a and the vertices in B with b . Therefore, $\chi(K_{m,n}) = 2$.
2. Let $f : V \rightarrow C$ be a coloring with $|C| = k$, and let $V_i = \{v \in V : f(v) = i\}$. Suppose $E \cap (V_i \times V_j) = \emptyset$ (there is no edge uv with $f(u) = i$ and $f(v) = j$), with $i \neq j \in C$. We define $f' : V \rightarrow C \setminus \{j\}$ as

$$f'(v) = \begin{cases} i & \text{if } v \in V_j \\ k & \text{if } v \in V_k, k \neq j \end{cases}.$$

Since $E \cap (V_i \times V_j) = \emptyset$, f' is a coloring of G with $|C \setminus \{j\}| = k - 1$. This is a contradiction to $\chi(G) = k$.

3. According to part 2., there is at least one edge for all $\{i, j\} \in \binom{[k]}{2}$. It follows that $|E| \geq \binom{k}{2}$. It follows that: $\binom{k}{2} \leq |E|$, which implies $\frac{k(k-1)}{2} \leq |E|$, and then $k^2 - k - 2|E| \leq 0$, leading finally to $k \leq \frac{1}{2} + \sqrt{\frac{1}{4} + 2|E|}$.

Exercise 2

[Chromatic polynomials]

1. Note that every coloring with k colors of G corresponds to a coloring with k colors of $G \setminus e$. On the other hand, every coloring $f : V(G \setminus e) \rightarrow [k]$ with $f(u) \neq f(v)$ gives a coloring of G .

Furthermore, every coloring $f : V(G \setminus e) \rightarrow [k]$ with $f(u) = f(v)$ corresponds exactly to one coloring of G/e . Thus, there is a bijection

$$\{\text{Colorings of } G \setminus e\} \longleftrightarrow \{\text{Colorings of } G\} \cup \{\text{Colorings of } G/e\}$$

and the statement follows.

2. **by Induction:** We show by induction on n that $P_{K_n}(k) = \prod_{i \in [n]} (k - i + 1)$ holds.

For $n = 1$ this is obviously true. Let $n \geq 2$. According to (i), we have

$$P_{K_n}(k) = P_{K_n \setminus e}(k) - P_{K_n/e}(k)$$

for any edge $e \in E(K_n)$. Furthermore, $K_n/e = K_{n-1}$, so by induction hypothesis

$$P_{K_n/e}(k) = \prod_{i \in [n-1]} (k - i + 1).$$

$K_n \setminus e$ is obtained from K_{n-1} by adding a new vertex v of degree $n - 2$. If a coloring $f : K_{n-1} \rightarrow [k]$ is chosen, it can be extended to a coloring $\tilde{f} : K_n \setminus e \rightarrow [k]$ by choosing $\tilde{f}(v)$ such that $\tilde{f}(v) \neq f(w)$ for all $w \in N(V)$. Thus, there are $k - (n - 2)$ ways to extend f and it follows that

$$P_{K_n \setminus e}(k) = (k - (n - 2))P_{K_{n-1}}(k)$$

and thus

$$P_{K_n}(k) = (k - (n - 2))P_{K_{n-1}}(k) - P_{K_{n-1}}(k) = (k - (n - 1))P_{K_{n-1}}(k) = \prod_{i \in [n]} (k - i + 1)$$

without Induction: For $k < n$, $P_{K_n}(k) = 0$.

For $k > n$, we choose an arbitrary order v_1, \dots, v_n of the vertices of K_n . There are k ways to color v_1 , then $k - 1$ ways for v_2 , $k - i + 1$ ways for v_i ($i \in [n]$). Therefore, it follows that $P_{K_n}(k) = \prod_{i \in [n]} (k - i + 1)$ (in fact, this equality also holds for $k < n$).

3. We proceed by induction on $m = |E|$:

If $m = 0$, then $G = (V, \emptyset)$ and each vertex $v \in V$ can be colored arbitrarily. Thus, $P_G(k) = k^n$.

Now let $m \geq 1$ and the statement hold for all graphs with at most $m - 1$ edges. Let $e \in E(G)$, then according to 1., $P_G(k) = P_{G \setminus e}(k) - P_{G/e}(k)$. By induction hypothesis, $P_{G \setminus e}(k)$ is a polynomial of degree n and $P_{G/e}(k)$ is a polynomial of degree $n - 1$. Overall, it follows that $P_G(k)$ is a polynomial of degree n .

Exercise 3

[Outerplanar graphs]

1. Since G is connected, we have $d(u) \geq 1$ for all $u \in V$. The only graph where all vertices have degree 1 consists of a single edge. Therefore, the statement follows.
2. Let G be outerplanar. We show the statement by induction on $|V|$. If $|V| \leq 3$, then each vertex can be colored with a different color and the statement follows. Now let $|V| \geq 4$. If G consists of multiple connected components, we can apply the induction hypothesis to each component. Thus, assume G is connected. According to 1., there exists a vertex u of degree ≥ 2 . If $d(u) = 2$, we can remove u . The resulting graph is still outerplanar and by induction hypothesis 3-colorable. Since the neighbors

of u use at most two colors, we can color u with one of the remaining colors. Therefore, assume $d(u) > 3$. Then there exists a neighbor v of u such that the edge uv is not on the boundary of the outerplanar drawing. This edge splits the drawing of G into two parts (such that no edge goes from one part to the other). Dividing G into these two parts gives rise to two outerplanar subgraphs, both containing uv and having fewer vertices. By induction hypothesis, these two graphs are 3-colorable and after possible recoloring, we can assume that u is colored the same in both graphs, and so is v . This way, we obtain a 3-coloring of G .

Exercise 4

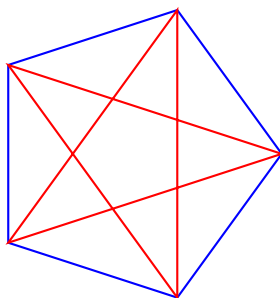
[Edge colorings]

1. By definition of an edge coloring, edges of the same color are not adjacent. Thus, edges of the same color form a matching. (Not necessarily a perfect matching, however.)
2. If G is r -regular, then $\Delta(G) = r$. If $\chi'(G) = \Delta(G) = r$, then for each vertex $u \in V(G)$ and each color $1 \leq i \leq r$, there is an edge uv with $c(uv) = i$. Here, c is a minimum edge coloring. According to 1., the edge set E_1 forms a matching of G . According to the previous argument, every vertex of V is covered.
3. From the lecture, we know that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$. If $\chi'(G) \neq \Delta(G) + 1$, then $\chi'(G) = \Delta(G)$ must hold, and according to (ii), G has a matching that covers V . Therefore, $|V|$ must be even.

Exercise 5

[Bonus – Ramsey theory]

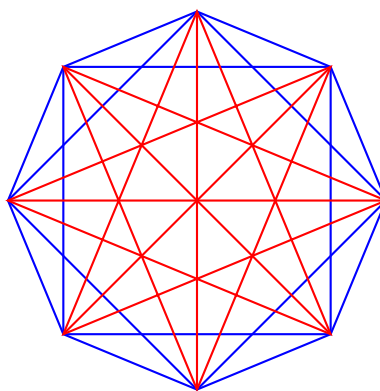
If $n = 5$, then the following coloring of the edges of K_5 leads to a draw.



Now, fix a red-and-blue coloring of the edges of K_n for $n \geq 6$. If $n = 6$, then fix a vertex v : it has degree $6 - 1 = 5$ in K_6 . Among these 5 neighbors, either at least 3 are connected to v by a blue edge or at least 3 are connected to v by a red edge. Suppose v has 3 blue-connected neighbors (the red case is symmetrical), if they are connected pair-wise by red edges, then there is a red triangle; otherwise, there are two blue-connected neighbors of v that are connected by a blue edge,

and thus we have a blue triangle. If $n \geq 6$, then K_n contains a K_6 for which any coloring will create a monochromatic triangle.

With $R(r, s) < R(r-1, s) + R(s, r-1)$ (see proof below), we obtain $R(3, 4) < R(2, 4) + R(3, 3)$. On the one hand, we have $R(3, 3) = 6$ by the above reasoning, and on the other hand, we have $R(2, n) = n$ because it is possible to color the edges of K_{n-1} avoiding creating a red K_n or a blue K_2 (by coloring all edges of K_{n-1} in red), but impossible to color the edges of K_n without making a blue edge K_2 or a red K_n . Hence, $R(3, 4) < 4 + 6 = 10$, i.e. $R(3, 4) \leq 9$. It remains to show that there exists an edge coloring of K_8 with no red K_3 nor blue K_4 . Here it is:



N.B. [Adapted from Wikipedia] We prove that $R(r, s) \leq R(r-1, s) + R(r, s-1)$ by induction on $r + s$.

Consider a complete graph on $n = R(r-1, s) + R(r, s-1)$ vertices whose edges are coloured in red and blue. Pick a vertex v from the graph, and partition the remaining vertices into two sets M and N , such that for every vertex w , w is in M if edge vw is blue, and w is in N if vw is red. Because the graph has $R(r-1, s) + R(r, s-1) = |M| + |N| + 1$ vertices, it follows that either $|M| \geq R(r-1, s)$ or $|N| \geq R(r, s-1)$. In the former case, if M has a red K_s then so does the original graph and we are finished. Otherwise M has a blue K_{r-1} and so $M \cup \{v\}$ has a blue K_r by the definition of M . The latter case is analogous. Thus the claim is true and we have completed the proof for 2 colours.

A strengthening of the argument gives the strict inequality when both $R(r-1, s)$ and $R(r, s-1)$ are even.

Exercise 6

[With ChatGPT]

ChatGPT is confusing between “graphs” and “graphs up to isomorphism”. When it counts 1024, that all graphs. When it count 6, that only graphs up to isomorphism. Hence, it subtract carrot to potatoes...

Up to isomorphism, there are 5 graphs on 5 vertices with $\chi(G) \geq 4$: take K_4 , add one vertex, and connect it to either 0, or 1, or 2, or 3, or 4 other vertices. A careful counting show that there are 958 graphs on 5 vertices with $\chi(G) \leq 3$, which gives 29 graphs up to isomorphism.

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 8

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Recall some notations on sequences of positive numbers:

$u_n = O(v_n)$ if there exists $C \geq 0$ such that $u_n \leq C v_n$ for all n large enough.

$u_n \sim v_n$ when $n \rightarrow +\infty$ if $\frac{u_n}{v_n} \rightarrow 1$ when $n \rightarrow +\infty$.

You can multiply (or divide) equivalent, i.e. if $u_n \sim v_n$ and $u'_n \sim v'_n$, then $u_n u'_n \sim v_n v'_n$ and $\frac{u_n}{u'_n} \sim \frac{v_n}{v'_n}$.

Exercise 1

[Hockey-stick identity]

We want to prove the following “hockey-stick identity” using different methods:

$$\text{for all } r \leq n, \quad \sum_{k=r}^n \binom{k}{r} = \binom{n+1}{r+1}$$

1. Using Pascal identity and a recursion on n , prove the hockey-stick identity.
2. Fix a crowd of $n+1$ persons.
 - (a) How many ways are there to form a group of $r+1$ people out of a crowd of $n+1$ people?
 - (b) Label $n-r+1$ of these persons (using the numbers $1, \dots, n-r+1$): for a given $k \in [1, n-r+1]$ how many ways are there to construct a group of $r+1$ persons whose minimum label is k ?
 - (c) Deduce the hockey-stick identity.
3. Let x be a symbol, and $X = 1 + x$.
 - (a) Prove that $X^r + \dots + X^n = \frac{1}{x}(X^{n+1} - X^r)$.
 - (b) Evaluate the coefficient on x^r on each side, and prove the hockey-stick identity.

Exercise 2

[Odd and distinct partitions]

For a fixed n , show that there are as many partitions $(\lambda_1, \dots, \lambda_m)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 1$ with all λ_i odd and $\sum_i \lambda_i = n$, that there are partitions (μ_1, \dots, μ_r) where $\mu_1 > \mu_2 > \dots > \mu_r \geq 1$ where all parts are distinct and $\sum_j \mu_j = n$.

To do so, one way is to exhibit a bijection between both types of partitions.

Hint: Split even parts into two parts, and use binary decomposition in a clever way.

Exercise 3

[Singmaster's theorem]

For $k \neq 1$, we want to estimate $N(k) = \#\{(n, r) ; \binom{n}{r} = k\}$. Especially, we want to prove Singmaster's theorem: $N(k) = O(\log k)$ when $k \rightarrow +\infty$.

1. Show that $N(k)$ is finite for $k \neq 1$, more precisely, show that $N(k) \leq \frac{k(k-1)}{2}$.
2. Look at the functions $b \mapsto \binom{a+b}{a}$ for fixed a , and $a \mapsto \binom{a+b}{a}$ for fixed b . Deduce that if s is such that $k \leq \binom{2s}{s}$, then $N(k) \leq 2s$.
3. Show that $2^m \leq \binom{2m}{m}$ for $m \geq 1$, and prove Singmaster's theorem.

N.B. A better estimation is known: $N(k) = O\left(\frac{\log \log \log k}{(\log \log k)^3} \log k\right)$, but it is conjectured that $N(k) = O(1)$, and the greatest known value of $N(k)$ is $N(3003) = 8$ (tested up to $k = 2^{48}$).

Exercise 4

[Useful bounds on binomial coefficients]

1. Show the left inequality in

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{n}{k}\right)^k e^k$$

2. Using that for all $k \geq 1$, we have $(1 + \frac{1}{k})^k \leq e$, show that $\frac{1}{k!} \geq (\frac{e}{k})^k$. Deduce the right side of the above inequality.
3. Fix a real number $c \in [0, \frac{1}{2}]$, and let $H(c) = -c \log_2 c - (1-c) \log_2 (1-c)$. Using that, when $n \rightarrow +\infty$, we have $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$, prove that, if $k = cn$, then $\binom{n}{k} \sim \frac{1}{\sqrt{2\pi c(1-c)n}} 2^{H(c)n}$.
4. Deduce the equivalent for Catalan numbers: $\frac{1}{n+1} \binom{2n}{n} \sim \frac{4^n}{n^{3/2} \sqrt{\pi}}$.

Exercise 5

[Bonus – Symmetric Venn diagram]

You draw a closed curve in the plane¹, and name S the region interior to this shape. You rotate this shape S by a n^{th} of a full turn, and paint each copy with its own color. Magically, you see, in front of your eyes, **all** the possible mixing of the colors you have used (see Figure 19).

Show that n is prime.

Submission: 18.06.2025 before 23:59

¹A closed curve is the image of a continuous function $f : [0, 1] \rightarrow \mathbb{R}^2$ with $f(0) = f(1)$, but you don't need that much formalism.

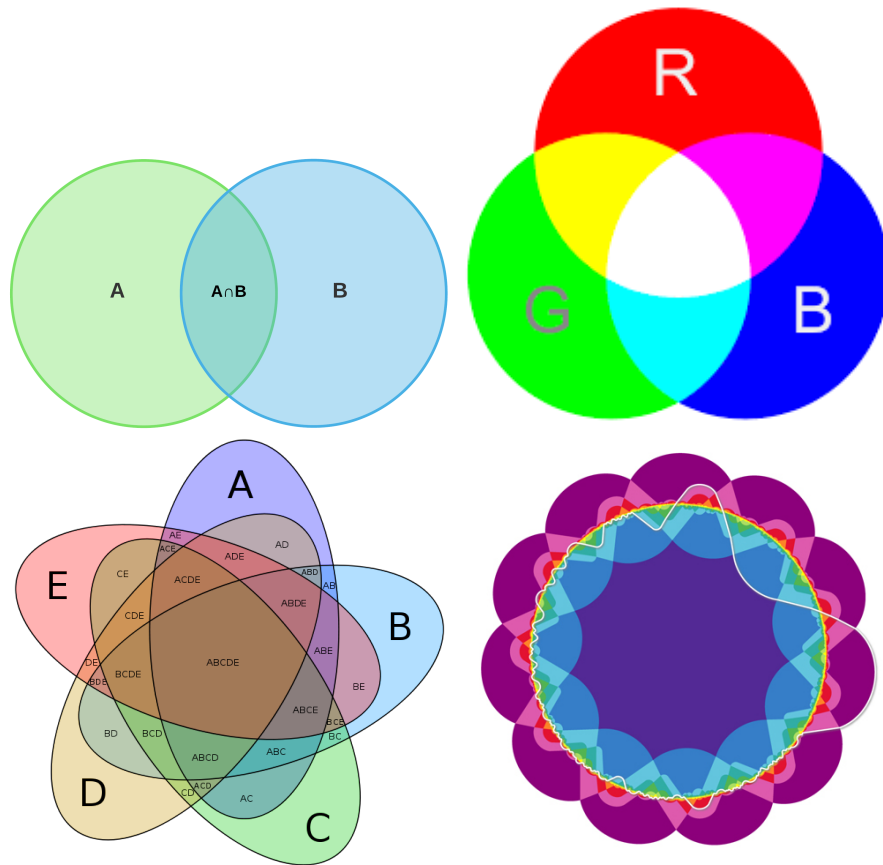


Abbildung 19: Symmetric Venn diagrams for $n \in \{2, 3, 5, 11\}$. The one for 11 is named *Newroz*, and was discovered in July 2012 (each region is painted violet, and the color changes according to the number of intersecting regions).

Tutorium additional exercises

Exercise 6

[With ChatGPT]

ChatGPT told me this (it is a real copy-paste from the free version of ChatGPT):

“ For a fixed n , we consider the set of binomial coefficients

$$S_n = \left\{ \binom{m}{k} \mid 0 \leq k \leq m \leq n \right\}$$

and define

$$a_n = \min (\mathbb{N} \setminus S_n)$$

as the smallest natural number that is *not* a binomial coefficient for any $m \leq n$.

We are interested in values of n such that

$$a_n = n + 4.$$

We find that for $n = 13$, the set of binomial coefficients S_{13} includes all natural numbers up to 16, but does *not* contain 17. Thus,

$$a_{13} = 17 = 13 + 4,$$

which satisfies the condition.

Therefore, $n = 13$ is the smallest such number for which $a_n = n + 4$. ”

This is false. Why?

Exercise 7

[Combinatorial identities]

Show, through combinatorial reasoning, that

$$\sum_{j=1}^{n-1} j(n-j) = \sum_{i=2}^n \binom{i}{2} = \binom{n+1}{3}.$$

Hint: Consider all subsets of $\{1, \dots, n+1\}$ with three elements and count them appropriately.

Exercise 8

[Chu-Vandermonde identity]

Let k, n, m be natural numbers.

1. Determine the coefficient of x^k in $(1+x)^{n+m}$.
2. Prove that

$$\sum_{j=0}^k \binom{n}{j} \binom{m}{k-j} = \binom{n+m}{k}$$

Hint: Look intensively at $(1+x)^m(1+x)^n = (1+x)^{n+m}$

3. Deduce $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$.

Exercise 9

[Chu-Vandermonde identity, version 2]

Prove Chu-Vandermonde identity (Exercise 3.2) by taking n red balls and m blue balls, and counting in two ways how many possibilities there are to pick k balls from this collection of $n+m$ balls.

Exercise 10 [Invertible matrices]
 Count the number of elements of $GL_n\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)$ for $n \geq 1$ and p prime.

Exercise 11 [Weak and Strong number compositions]
 Let $n, r \geq 1$ be natural numbers. A *weak r -composition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of natural numbers $\lambda_i \geq 0$, such that $n = \lambda_1 + \dots + \lambda_r$ (note that λ_i are not ordered). If $\lambda_i \geq 1$ for $1 \leq i \leq r$, then λ is called a *strong r -composition* of n .

1. How many weak and strong 2-compositions of n are there?
2. Show that the number of weak r -compositions of $n - r$ is equal to the number of strong r -compositions of n .

Exercise 12 [Trinomial theorem]

1. Let $a_{n,i,j}$ be the number of words of length n over the alphabet $\{x, y, z\}$ that contain exactly i occurrences of the letter x , and j occurrences of the letter y . For example:

$$a_{4,1,1} = |\{xyzxz, xzyxz, xzzzy, yxzzz, yzxxz, yzzzx, zxyxz, zxzyz, zzzxy, zyxzx, zyzxz, zzyxz\}| = 12$$

Show that $a_{n,i,j} = \frac{n!}{i!j!(n-i-j)!}$.

2. Prove the *Trinomial Theorem*:

$$(x + y + z)^n = \sum_{0 \leq i+j \leq n}^n a_{n,i,j} x^i y^j z^{n-i-j}.$$

Hint: Consider $w = x + y$. Then $(x + y + z)^n = (w + z)^n$ and you can use the Newton's Binomial theorem.

3. Derive from 2. two different expressions for the number of words of length n over the alphabet $\{x, y, z\}$.

Exercise 13 [Binomial coefficients and congruence]
 Let p be a prime number. Show that $\binom{p}{k}$ is divisible by p , except if $k \in \{0, p\}$. Deduce that $(a + b)^p - (a^p + b^p)$ is divisible by p , for any integers a and b .

Discrete Mathematics – Solutions Sheet 8

Exercise 1

[Hockey-stick identity]

1. If $r = n$, the identity is trivial (both sides are 1). Suppose the identity hold for some n and for any $r \leq n$, then we have:

$$\sum_{k=r}^{n+1} \binom{k}{r} = \binom{n+1}{r} + \sum_{k=r}^n \binom{k}{r} = \binom{n+1}{r} + \binom{n+1}{r+1} = \binom{n+2}{r+1}$$

This concludes the proof.

- (a) There are $\binom{n+1}{r+1}$ ways to make a group of $r+1$ persons out of a crowd of $n+1$ persons.
 - (b) If k is the smallest label in the group, then there is no-one with label between 1 and $k-1$, hence it amounts to choosing a group of r persons in a crowd of $n-(k-1)$ persons (then adding the person labeled k): there are $\binom{n-k+1}{r}$ ways.
 - (c) There is a bijection between the ways of making a group of $r+1$ persons in a crowd of $n+1$ persons, and the couples $(k, \text{a way to make a group of } r+1 \text{ persons in a crowd of } n+1 \text{ persons where the minimum of the group is } k)$. To see that this is a bijection, note that any group contains a labeled person (because there are only r non-labeled persons), and that a group has a unique minimum. Hence, via this bijection, we get the hockey-stick identity.
2. (a) Via the well-known geometric summation: $X^r + \dots + X^n = \frac{X^r - X^{n+1}}{1-X} = \frac{1}{x}(X^{n+1} - X^r)$
 - (b) On the left side, the coefficient on x^r of $X^k = (1+x)^k$ is $\binom{k}{r}$, via Newton binomial formula. On the right side, the coefficient on x^r is the one on x^{r+1} of $(1+x)^{n+1} - (1+x)^r$, which is $\binom{n+1}{r+1} - 0$. Hence, we get the hockey-stick identity (again).

Exercise 2

[Odd and distinct partitions]

Take a partition of n into distinct parts $\mu_1 > \mu_2 > \dots > \mu_r \geq 1$. For each even part $\mu_j = 2x$, replace it by two parts of value x . If every part is odd, we have obtained a partition into odd parts, else we can repeat the process in order to obtain a partition of n containing only odd parts. We have a map f from distinct partitions to odd partitions.

Let's construct the reciprocal of f . Take $\lambda_1 \geq \dots \geq \lambda_m \geq 1$ a partition of n into odd parts. Suppose λ_j is repeated p times (meaning that $\lambda_j = \lambda_{j'}$ for p different values of j'), with possibly $p = 1$. Write $p = \sum_{i \in I} 2^i$ (the binary decomposition of p), and replace the p copies of λ_j by $\mu_i = \lambda_j \times 2^i$ for $i \in I$. Reordering μ give rises to a partition into distinct parts because all integer numbers have a **unique** way to be written as $k \times 2^i$ with k odd (just take the prime decomposition of your number and isolate the power of 2).

A quick scribble shows that these two constructions are reciprocal from one another, giving rise to a bijection between odd and distinct partitions (hence, there are the same number of both).

Exercise 3

[Singmaster's theorem]

1. If $n > k$, then $\binom{n}{r} > k$ for all $r \in \{1, \dots, n-1\}$, and moreover, $\binom{n}{0} = \binom{n}{n} = 1 \neq k$ for all n . Hence, the possible values of n and r such that $\binom{n}{r} = k$ are limited into the (interior of the) part of Pascal's triangle that lies above the k -th line, that is to say $n \leq k$ and $1 \leq r \leq n-1$. There are $\frac{k(k-1)}{2}$ such couples (n, r) , which proves $N(k) \leq \frac{k(k-1)}{2} < +\infty$.
2. The maps $f_a : b \mapsto \binom{a+b}{a}$ for fixed a , and $f_b : a \mapsto \binom{a+b}{b}$ for fixed b are strictly increasing. Consequently, if $\binom{a+b}{b} = k$ for some b and for a fixed, then $\binom{a+b}{a} \neq k$ for all other b (and same a); and the same holds symmetrically. Now, if $k \leq \binom{2s}{s}$, then if $k = \binom{a+b}{a}$, then $a \leq s$ and $b \leq s$ (because f_a and f_b are increasing). Moreover, each choice of a give rise to at most one possible solution for $\binom{a+x}{a} = k$, and each choice of b also, so the number of solutions is at most: (the number of possible a) + (the number of possible b) $\leq s + s = 2s$, i.e. $N(k) \leq 2s$.

3. For $m \geq 1$:

$$\begin{aligned}
 (2m)! &= \prod_{k=1}^{2m} k \\
 &= \prod_{k=1}^m (2k) \prod_{k=0}^{m-1} (2k+1) \\
 &\geq 2^m \prod_{k=1}^m k \prod_{k=0}^{m-1} k \\
 &\geq 2^m (m!)^2
 \end{aligned}$$

Hence, $2^m \leq \frac{(2m)!}{(m!)^2} = \binom{2m}{m}$, and thus, for a fixed k the smallest s such that $k \leq \binom{2s}{s}$ satisfies $\binom{2(s-1)}{s-1} \leq k$, so $2^{s-1} \leq k$, implying $s \leq 1 + \log_2 k$. Finally, $N(k) \leq 2s \leq 2 + \frac{2}{\log 2} \log k = O(k)$.

Exercise 4

[Useful bounds on binomial coefficients]

1. A quick scribble gives $\frac{n}{k} \leq \frac{n-x}{k-x}$ for all $x \in [1, k-1]$. Hence:

$$\binom{n}{k} = \frac{n}{k} \frac{n-1}{k-1} \frac{n-2}{k-2} \cdots \frac{n-k+1}{1} \geq \frac{n}{k} \frac{n}{k} \frac{n}{k} \cdots \frac{n}{k} = \left(\frac{n}{k}\right)^k$$

2. For the upper bound, note that $\frac{(k+1)^{k+1}}{(k+1)!} = \frac{(k+1)^k}{k!} = \left(\frac{k+1}{k}\right)^k \frac{k^k}{k!} \leq e \frac{k^k}{k!}$. By induction: $\frac{1}{k!} \leq \frac{e^k}{k^k}$. Consequently, $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \leq \frac{n^k}{k!} \leq \left(\frac{n}{k}\right)^k e^k$.

3. Using the Stirling equivalent $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$, we get:

$$\begin{aligned}
\binom{n}{cn} &= \frac{n!}{(cn)!((1-c)n)!} \\
&\sim \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{\left(\frac{cn}{e}\right)^{cn} \sqrt{2\pi cn} \left(\frac{(1-c)n}{e}\right)^{(1-c)n} \sqrt{2\pi(1-c)n}} \\
&\sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{c(1-c)}} \frac{1}{\sqrt{n}} \left(\frac{1}{c^c(1-c)^{1-c}}\right)^n \\
&\sim \frac{1}{\sqrt{2\pi c(1-c)n}} 2^{H(c)n}
\end{aligned}$$

4. We have $H\left(\frac{1}{2}\right) = 1$. Hence, substituting n by $2n$ in the previous formula, and c by $\frac{1}{2}$, we get: $\frac{1}{n+1} \binom{2n}{n} \sim \frac{1}{n} \frac{1}{\sqrt{2\pi \frac{1}{4} 2n}} 2^{2n}$, which reduce to the desired form.

Exercise 5

[Bonus – Symmetric Venn diagram]

Consider the colored sub-regions that correspond to the mixing of k different colors. On the one hand, there are $\binom{n}{k}$ such sub-regions, because each region correspond to any choice of k colors among the n colors used. Besides, as the drawing is symmetric, one can start with a sub-region and consider all the other sub-regions one can obtain by (multiple) rotations of a n -th of a full turn. This partitions the collection of sub-regions into n parts of the same size, hence: $n \mid \binom{n}{k}$.

This means that $\frac{1}{n} \binom{n}{k}$ is a integer for all k , especially for a prime divisor p of n with $p \neq n$. But: $\frac{1}{n} \binom{n}{p} = \frac{(n-1)(n-2)\dots(n-p+1)}{p!}$. The denominator of this fraction has p as a factor, but the numerator has not (as $p \mid n$, we know that p does not divide any $j \in \{n-p+1, \dots, n-1\}$), so the right-hand-side is not a integer because p is co-prime with $(n-1)(n-2)\dots(n-p+1)$.

Consequently, n is prime (he has no other prime divisor than itself).

Exercise 6

[With ChatGPT]

ChatGPT doesn't know how to count: $a_{13} = 14$, as 14 is not a binomial coefficient for $n \leq 13$ (print a big Pascal triangle). Actually, I didn't find any n with $a_n = n + 4$, for $n \leq 300$.

Exercise 7

[Combinatorial Identities]

We aim to show that

$$\sum_{j=1}^{n-1} j(n-j) = \sum_{i=2}^n \binom{i}{2} = \binom{n+1}{3}.$$

To demonstrate this identity, we note that all three expressions count the same quantity. First, from Theorem 1.3 (c), we know that $\binom{n+1}{3}$ counts the 3-element subsets of $[n+1]$. We will show that the other two expressions can be interpreted similarly.

First, let's classify the 3-element subsets of $[n+1]$ by their middle element. Let $A = \{i, j, k\} \subseteq [n+1]$ with $i < j < k$. Then, it holds that $2 \leq j \leq n$. We denote by M_j the set of 3-element subsets of $[n+1]$ with the middle element j , where $2 \leq j \leq n$. We now determine the number of elements in M_j . To specify a set A in M_j , we need to choose the smallest and largest element of A . For the smallest element, there are $j-1$ possibilities, and for the largest, there are $(n+1)-j$ possibilities. By the product rule, it follows that $|M_j| = (j-1)(n+1-j)$. Since $\binom{[n+1]}{3} = \cup_{j=2}^n M_j$, it follows by the sum rule that

$$\binom{n+1}{3} = \left| \binom{[n+1]}{3} \right| = \sum_{j=2}^n |M_j| = \sum_{j=2}^n (j-1)(n+1-j) = \sum_{j=1}^{n-1} j(n-j).$$

This shows one of the equalities to be proven.

For the second equality, we classify the 3-element subsets of $[n+1]$ according to their largest element. Let $A = \{i, j, k\} \subseteq [n+1]$ with $i < j < k$. Then, $3 \leq k \leq n+1$. Let N_k be the set of 3-element subsets of $[n+1]$ with the largest element k , where $3 \leq k \leq n+1$. We now determine the number of elements in N_k . To specify a set A in N_k , we need to choose the smallest and middle element. Since k is the largest element of A , there are $\binom{k-1}{2}$ possibilities for the other two elements. Therefore, $|N_k| = \binom{k-1}{2}$. Since $\binom{[n+1]}{3} = \cup_{k=3}^{n+1} N_k$, it follows by the sum rule that

$$\binom{n+1}{3} = \left| \binom{[n+1]}{3} \right| = \sum_{k=3}^{n+1} |N_k| = \sum_{k=3}^{n+1} \binom{k-1}{2} = \sum_{i=2}^n \binom{i}{2}.$$

This shows the second equality.

Exercise 8

[Chu-Vandermonde identity]

1. According to Newton's binomial formula, we have $(1+x)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} x^k$, so the coefficient on x^k is $\binom{n+m}{k}$.

2. On the other side:

$$\begin{aligned} (1+x)^{n+m} &= (1+x)^n (1+x)^m \\ &= \left(\sum_{i=0}^n \binom{n}{i} x^i \right) \left(\sum_{j=0}^m \binom{m}{j} x^j \right) \\ &= \sum_{k=0}^{n+m} \left(\sum_{i+j=k} \binom{n}{i} \binom{m}{j} \right) x^k \end{aligned}$$

So, by identification of the coefficient, we get Chu-Vandermonde formula: $\sum_{i+j=k} \binom{n}{i} \binom{m}{j} = \binom{n+m}{k}$ (replace i by $k-j$). A good exercise is also to find a combinatorial proof of this identity.

Exercise 9

[Chu-Vandermonde identity, version 2]

On the one side, there are $\binom{n+m}{k}$ ways to pick k balls from a collection of $n+m$ balls. On the other side, picking k balls in a collection of $n+m$ balls amounts to picking j balls in a collection of n balls and picking $k-j$ balls in a collection of m , where j can be any integer from 0 to k . Seeing the problem in this fashion, there are $\sum_{j=0}^k \binom{n}{j} \binom{m}{k-j}$ way to pick k balls in a collection of $n+m$ balls.

Exercise 10

[Invertible matrices]

A matrix in $GL_n(\frac{\mathbb{Z}}{p\mathbb{Z}})$ is just a collection of n vectors that are linearly independent: (v_1, \dots, v_n) .

When $n = 1$, there are $p^n - 1$ possibilities because each of the n coordinate of v_1 can be any number from 0 to $p-1$, but $v_1 \neq \mathbf{0}$.

Pick any $(v_1, \dots, v_k, v_{k+1})$ linearly independent (for $k+1 \leq n$). Having independence is equivalent to say that: (v_1, \dots, v_k) is linearly independent, and v_{k+1} cannot be written as $\sum_{j=1}^k a_j v_j$, with $a_j \in \{0, \dots, p-1\}$. Hence, for each choice of independent (v_1, \dots, v_k) , there are $n^p - p^k$ valid choices for v_{k+1} (n^p vectors in $(\frac{\mathbb{Z}}{p\mathbb{Z}})^n$, among which p^k are forbidden).

Consequently, $\#GL_n(\frac{\mathbb{Z}}{p\mathbb{Z}}) = \prod_{k=0}^{n-1} (n^p - p^k)$.

Exercise 11

[Weak and Strong Number compositions]

1. The strong 2-compositions of n are given by $(i, n-i)$ for $1 \leq i \leq n-1$, i.e., there are $n-1$ of them. For weak 2-compositions, the compositions $(0, n)$ and $(n, 0)$ are additionally included, i.e., there are $n+1$ of them.
2. We need to show that the number of weak r -compositions of $n-r$ is equal to the number of ordered r -compositions of n . To this end, let

$$\mathcal{P}_1 := \{\lambda = (\lambda_1, \dots, \lambda_r) : \lambda_i \geq 0 \text{ for } 1 \leq i \leq r, \lambda_1 + \dots + \lambda_r = n-r\}$$

be the set of weak r -compositions of $n-r$ and let

$$\mathcal{P}_2 := \{\lambda = (\lambda_1, \dots, \lambda_r) : \lambda_i \geq 1 \text{ for } 1 \leq i \leq r, \lambda_1 + \dots + \lambda_r = n\}$$

be the set of r -compositions of n . We consider the following mapping

$$\begin{aligned} \mathcal{P}_1 &\rightarrow \mathcal{P}_2 \\ \lambda &= (\lambda_1, \dots, \lambda_r) \mapsto \Phi(\lambda) = (\lambda_1 + 1, \dots, \lambda_r + 1). \end{aligned}$$

We first show that the mapping Φ is *well-defined*. Let $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}_1$. Since $\lambda_i \geq 0$, it follows that $\lambda_i + 1 \geq 1$ for $1 \leq i \leq r$. And since $\lambda_1 + \dots + \lambda_r = n-r$, it also holds that $(\lambda_1 + 1) + \dots + (\lambda_r + 1) = (n-r) + r = n$. Thus, $\Phi(\lambda)$ is an r -composition of n , i.e., $\Phi(\lambda) \in \mathcal{P}_2$.

Φ is also *injective*, because if $\lambda, \mu \in \mathcal{P}_2$ such that

$$\Phi(\lambda) = (\lambda_1 + 1, \dots, \lambda_r + 1) = (\mu_1 + 1, \dots, \mu_r + 1) = \Phi(\mu),$$

then it directly follows that $\lambda_i = \mu_i$ for $1 \leq i \leq r$, i.e., $\lambda = \mu$.

Furthermore, Φ is *surjective*. For $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}_2$, it follows as in the proof of well-definedness that $\lambda' := (\lambda_1 - 1, \dots, \lambda_r - 1) \in \mathcal{P}_1$ holds and additionally, $\Phi(\lambda') = \lambda$.

Thus, Φ is a bijection and by the rule of equality, it follows that $\#\mathcal{P}_1 = \#\mathcal{P}_2$.

Exercise 12

[Trinomial Theorem]

1. The number $a_{n,i,j}$ of words of length n in the alphabet $\{x, y, z\}$ that contain exactly i x 's and j y 's is:

$$\begin{aligned} & |\{i\text{-element subset of } [n]\}| \cdot |\{j\text{-element subset of } [n-i]\}| = \\ & \binom{n}{i} \binom{n-i}{j} = \\ & \frac{n!}{i!(n-i)!} \frac{(n-i)!}{j!(n-i-j)!} = \frac{n!}{i!j!(n-i-j)!}. \end{aligned}$$

- 2.

$$\begin{aligned} (x + y + z)^n &= (x + y + z)^n = \\ &= \sum_{i=0}^n \binom{n}{i} x^i (y + z)^{n-i} = \\ &= \sum_{i=0}^n \binom{n}{i} x^i \sum_{j=0}^{n-i} \binom{n-i}{j} y^j z^{n-i-j} = \\ &= \sum_{0 \leq i+j \leq n} \binom{n}{i} \binom{n-i}{j} x^i y^j z^{n-i-j} = \\ &= \sum_{0 \leq i+j \leq n} a_{n,i,j} x^i y^j z^{n-i-j}. \end{aligned}$$

3. Let $A_{n,i,j}$ be the set of words of length n in the alphabet $\{x, y, z\}$ that contain exactly i x 's and j y 's.

$$\begin{aligned} & |\text{Words of length } n \text{ in the alphabet } \{x, y, z\}| = \\ & \sum_{0 \leq i+j \leq n} |A_{n,i,j}| = \\ & \sum_{0 \leq i+j \leq n} a_{n,i,j} = \\ & (1 + 1 + 1)^n = 3^n. \end{aligned}$$

Exercise 13

[Binomial coefficients and congruence]

Firstly, $\binom{p}{1} = p$ is divisible by p .

Suppose $\binom{p}{k}$ is divisible by p for some $1 \leq k \leq p-2$. Then, $\binom{p}{k+1}(k+1) = \binom{p}{k}(p-k-1)$. As p divides the right-hand-side, it also divides the left-hand-side. As p is co-prime with $k+1$ (because p is prime and $0 < k+1 < p$), we get that p divides $\binom{p}{k+1}$. By induction, p divides all $\binom{p}{k}$ for $1 \leq k \leq p-1$.

This implies that $(a+b)^p = a^p + b^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k b^{p-k} \equiv a^p + b^p \pmod{p}$.

This only works because p is prime!

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 9

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1

[Permutations of Multisets]

Let $A = \{a_1, \dots, a_\ell\}$ be a set and M the multiset over A where the element a_i occurs with multiplicity k_i .

1. Show that if A and B are sets and $\varphi : A \rightarrow B$ is a surjective map, then $|A| = \sum_{b \in B} |\varphi^{-1}(\{b\})|$
2. Construct a surjective map $\varphi : [n]^{\langle n \rangle} \rightarrow P_M$ with $|\varphi^{-1}(\{\pi\})| = \prod_{i=1}^{\ell} k_i!$
3. Conclude that $|P_M| = \frac{n!}{\prod_{i=1}^{\ell} k_i!}$
4. How many anagrams of the word DISCRETEMATHEMATICS exist (even if they don't make sense) ?

Exercise 2

[Inclusion-exclusion inequalities]

Let A_1, \dots, A_n be finite sets.

1. Show that: $\sum_i |A_i| - \sum_{i \neq j} |A_i \cap A_j| \leq |\bigcup_{i=1}^n A_i| \leq \sum_i |A_i|$
2. Using a recursion on K , show that, for $K \leq m$, we have (with $\binom{m}{m+1} = 0$):

$$\sum_{k=1}^K (-1)^{k+1} \binom{m}{k} = 1 - (-1)^K \binom{m-1}{K}$$

3. Show that, if $K \in [n]$ is odd, then:

$$\left| \bigcup_{i=1}^n A_i \right| \leq \sum_{k=1}^K (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} |A_{i_1} \cap \dots \cap A_{i_k}|$$

And show that, if $K \in [n]$ is even, the inequality is reversed.

4. For some fixed K , show that one the previous inequality holds is actually an equality if and only if the intersection of any $K+1$ sets chosen among the sets A_1, \dots, A_n is empty.

Exercise 3

[Pocket money with restrictions]

You have 30 coins of 1€. You want to distribute them to your 3 children.

1. Show that the number of ways to distribute 30 coins of 1€ among your 3 children is the coefficient on X^{30} in $P(X) = (1 + X + X^2 + \cdots + X^{30})^3$.
2. You now want each child to get at least 1€: modify the polynomial P so that the number of ways to do so is the coefficient on X^{30} of your new polynomial.
3. You want that the oldest child gets at least 10€: modify the polynomial P so that the number of ways to do so is the coefficient on X^{30} of your new polynomial.
4. You want that the 2 youngest children gets at least 8€: modify the polynomial P so that the number of ways to do so is the coefficient on X^{30} of your new polynomial.
5. How many ways are there to distribute 30 coins of 1€ among your 3 children (without any restriction)?

Exercise 4 [Radius of convergence and asymptotic behavior]

Let $(a_n)_{n \geq 0}$ be a sequence of non-negative (integer) numbers, and define the series $A(z) = \sum_{n \geq 0} a_n z^n$. We want to estimate the asymptotic growth of a_n when $n \rightarrow +\infty$ from “easy to get” properties of $A(z)$. In this exercise, $z \in \mathbb{R}$.

We define the *radius* of the sequence $(a_n)_{n \geq 0}$ as the biggest r such that $a_n \leq (\frac{1}{r})^n$ for any n big enough (i.e. there exists n_0 such that for all $n \geq n_0$, we have $a_n \leq (\frac{1}{r})^n$).

1. For the sequence $a_n = 5^n$, compute the radius r , compute the generating series $A(z)$, and show that $\sum_{n \geq 0} a_n z^n \rightarrow +\infty$ if $z > r$; whereas $\sum_{n \geq 0} a_n z^n \not\rightarrow +\infty$ if $z < r$.
2. Fix a sequence $(a_n)_{n \geq 0}$ with radius r . Show that $\sum_{n \geq 0} a_n z^n \rightarrow +\infty$, for any $z > r$.
3. Conversely, show that $\sum_{n \geq 0} a_n z^n \not\rightarrow +\infty$, for any $z < r$.

Hint: Pick n_0 cleverly, then multiply the term of the series by $\frac{r^n}{r^n}$.

The radius is an approximation of the exponential growth of a sequence: if the sequence $(a_n)_{n \geq 0}$ has radius r , then one may think that a_n is roughly equal to r^n when $n \rightarrow +\infty$ (usually, $a_n = F(n) r^n$ where F is a rational fraction, but this should not be regarded as a rigorous mathematical statement).

You have just proven that the radius of the sequence $(a_n)_{n \geq 0}$ is equal to “the smallest z_0 such that either $A(z_0)$ is not well-defined, or such that $A(z)$ is not well-defined for any $z = z_0 + \varepsilon$ (for all $\varepsilon > 0$ small enough)”. This is not mathematically formal, but you can use it for the next question (and in life).

4. Complete the following table (if you do not complete the last 2 lines, you will not loose points, neither will you get bonus points for doing them):

Sequence a_n	Series equation	r^n
Exponential ρ^n for some $\rho > 0$	$A(z) = \frac{1}{1-\rho z}$	
Fibonacci numbers	$A(z) = \frac{z}{1-z-z^2}$	
Catalan numbers	$A(z) = \frac{1-\sqrt{1-4z}}{2z}$	
Motzkin numbers	$A(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$	
Schröder numbers	$A(z) = \frac{1-z-\sqrt{1-6z+z^2}}{2z^2}$	
Cayley numbers	$A(z) = ze^{A(z)}$	
Rooted trees with 2 or 3 children	$\frac{A(z)}{1+A(z)^2+A(z)^3} = z$	

Exercise 5

[Bonus – Encoding a set through its sum]

A student is trying to implement something. A set of 10 starting numbers is given, all of them being 2-digit numbers. From time to time, the user will give a subset of these numbers, and the student has to store it in the memory of his computer, in order to give it back to the user later. To spare memory, the student decides to implement the following method: he will only store the sum of the numbers that the user gives him (and the student hopes he will find an algorithm to retrieve the whole subset from its sum).

Show that there is no hope for this method to work, whatever the 10 starting numbers.

Show that this method works if the starting numbers are $\{1, 2, 4, 8, 16, 32, 64\}$.

Submission: 25.06.2025 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

Exercise 7

[Divisibility of sums]

Fix k integers (not necessarily positive) n_1, \dots, n_k . Show that there is $X \subseteq [k]$ such that the sum $\sum_{i \in X} n_i$ is divisible by k .

Hint: Start by looking at $n_1 + \dots + n_j$ for each $j \in [k]$ and wonder what happens if none of these sums is divisible by k .

Exercise 8

[Fleissner's grids]

Rotating grids, developed by Colonel Fleissner, were used in a cryptography method used by the Germans during the First World War. Such a grid consists of a square with a side length of 6. This square is divided into a grid of 36 smaller equal squares (all with a side length of 1), and 9 of these squares are removed. The following property must be verified: the holes obtained with the grid in its initial position, with the grid rotated a quarter turn, a half turn, or three-quarter turn, never overlap. Thus, all 36 positions can be occupied by a hole after possibly rotating the grid a quarter, half, or three-quarter turn (and will be occupied only once).

1. How many such grids can be made?
2. For which values of n can a Fleissner's $n \times n$ grid be made? How many such grids can be made?

Exercise 9

[Triangles from points in the plane]

Let $n > 0$ be an integer. Let S be a set of n points in the plane such that any three distinct points of S are not aligned. Show that there are at most $\frac{1}{3}n(n-1)$ equilateral triangles formed by three distinct points in S .

Hint: Double-count the correct set of pairs.

Exercise 10

[Points and circles]

Let n and k be two positive integers. Let S be a set of n points in the plane such that any three distinct points of S are not aligned, and for every point $\mathbf{p} \in S$, there exists a circle with center \mathbf{p} that contains at least k points of S . Show that $k < \frac{1}{2} + \sqrt{2n}$.

Discrete Mathematics – Solutions Sheet 9

Exercise 1

[Permutations of Multisets]

1. Let $b \in B$. Since φ is surjective, there exists $a \in A$ with $\varphi(a) = b$. Furthermore, $\varphi(a)$ is uniquely determined. Therefore, $\bigcup_{b \in B} \varphi^{-1}(b) = A$, and $\varphi^{-1}(b) \cap \varphi^{-1}(b') = \emptyset$ for $b \neq b'$. Hence, the claim follows.
2. We define $\varphi : [n]^{(n)} \rightarrow P_M$, $\sigma \mapsto \varphi(\sigma)$ with $\varphi(\sigma)(m_i) = m_{\sigma(i)}$. Obviously, φ is well-defined and surjective. Let $m \in M$ be an element with multiplicity k . For $\pi \in P_M$ and $\sigma, \sigma' \in \varphi^{-1}(\pi)$, it holds that $\sigma' = \sigma \circ \tau_1 \circ \dots \circ \tau_\ell$, where τ_i is a permutation of $\{\sum_{j < i} k_j + 1, \dots, \sum_{j \leq i} k_j\}$. Since there are exactly $k_i!$ such permutations for each i , the claim follows.
3. Combining 1. and 2. gives:

$$n! = \left| [n]^{(n)} \right| = \left| \sum_{\pi \in P_M} \varphi^{-1}(\{\pi\}) \right| = |P_M| \prod_{i=1}^{\ell} k_i!$$

4. An anagram (if we don't care about meaning) is just a permutation of the letters of the word. Hence, for the word DISCRETEMATHEMATICS, we write the letters as a multiset. There are 18 letters, more precisely 1 A, 2 C, 1 D, 3 E, 1 H, 2 I, 2 M, 1 R, 2 S, 3 T. Consequently, the number of anagrams of this word is: $\frac{18!}{1!2!1!3!1!2!2!1!2!3!} = \frac{18!}{2^4 6^2} = 11\,115\,232\,128\,000$.

Exercise 2

[Inclusion-exclusion inequalities]

1. Consider $X = \{(a, i) ; i \in [n], a \in A_i\}$. Clearly, $|X| = \sum_i |A_i|$, and there is an injection $X \rightarrow \bigcup_i A_i$ given by $(a, i) \mapsto a$. Hence, $|\bigcup_i A_i| \leq \sum_i |A_i|$. On the other side, consider $a \in \bigcup_i A_i$. If a belong to exactly 1 of the A_i s, then it is counted once in the sum $\sum_i |A_i|$ but not counted in the sum $\sum_{i \neq j} |A_i \cap A_j|$. More generally, if a belong to several of the A_i s, say $k(a)$ of them, then it is counted once in $\sum_i |A_i|$, and $\binom{k(a)}{2}$ times in the sum $\sum_{i \neq j} |A_i \cap A_j|$. Denoting $\binom{1}{2} = 0$, we get $\sum_i |A_i| - \sum_{i \neq j} |A_i \cap A_j| = \sum_{a \in \bigcup_i A_i} 1 - \binom{k(a)}{2} \leq \sum_{a \in \bigcup_i A_i} 1 = |\bigcup_i A_i|$.
2. Fix m . For $K = 1$, we have that: $(-1)^2 \binom{m}{1} = m = 1 - (-1)^1 \binom{m-1}{1}$. Recall Pascal's identity (written weirdly): $\binom{m}{K+1} - \binom{m-1}{K} = \binom{m-1}{K+1}$. Suppose the

formula holds for some K , then:

$$\begin{aligned}\sum_{k=1}^{K+1} (-1)^{k+1} \binom{m}{k} &= (-1)^{K+2} \binom{m}{K+1} + \sum_{k=1}^K (-1)^{k+1} \binom{m}{k} \\ &= (-1)^K \binom{m}{K+1} + 1 - (-1)^K \binom{m-1}{K} \\ &= 1 - (-1)^{K+1} \binom{m}{K+1}\end{aligned}$$

3. Fix $K \in [n]$. The right-hand-side is the usual term in the inclusion-exclusion principal, except that the sum stops at K instead of n . As for the proof of the inclusion-exclusion principal, we consider each $x \in \bigcup_i A_i$, and count its mutiplicity in the right-hand sum.

Fix $x \in \bigcup_i A_i$. For $T \subseteq [n]$, the element x contribute to the right-hand sum if and only if $x \in A_T := \bigcap_{i \in T} A_i$. In the right-hand sum, we only consider $T \subseteq [n]$ with $|T| \leq K$, hence the total contribution of x to the right-hand sum is $\sum_{T \subseteq [n], |T| \leq K, x \in A_T} (-1)^{|T|+1}$. Let m be the number of indices i such that $x \in A_i$ (i.e. $m := |\{i \in [n] ; x \in A_i\}|$), then there are $\binom{m}{k}$ sets T of size k such that $x \in A_T$ (for $k \geq 1$). Consequently, the contribution of x rewrites as $\sum_{k=1}^{\min(m, K)} (-1)^{k+1} \binom{m}{k}$. If $\min(m, K) = m$, then this contribution is 1 (see course); else, according to question 2., is it $1 - (-1)^K \binom{m-1}{K}$. The later is bigger than 1 if K is even, and small than 1 if K is odd. Thus, the right-hand sum is bigger than $\sum_{x \in \bigcup_i A_i} 1 = |\bigcup_i A_i|$ if K is even, and smaller if K is odd. This gives the claimed inequalities.

4. To have an equality case, we need that, for all x , the minimum between K and m (recall that m is the number of indices i such that $x \in A_i$) is m . Equivalently, we need that there is no x which belong to (strictly) more than K different A_i . This is equivalent to the announced sentence.

Exercise 3

[Pocket money with restriction]

1. A way to distribute 30 coins of 1€ among 3 children is a equivalent to an (ordered) partition of 30 into 3 parts. You can write the polynomial form directly from the course. Or, you can develop:

$$\left(\sum_{a=0}^{30} X^a \right)^3 = \sum_{a,b,c=0}^{30} X^a X^b X^c = \sum_{k=0}^{90} \left(\sum_{a+b+c=k} 1 \right) X^k$$

The number $\sum_{a+b+c=k} 1$ is the number of ways to partition a set of k elements into 3 parts (where each part can have between 0 and 30 elements).

2. Use the polynomial $(X + \dots + X^{30})^3$, and re-read the above reasoning, replacing 0 by 1.
3. Use the polynomial $(1 + X + \dots + X^{30})^2 \cdot (X^{10} + X^{11} + \dots + X^{30})$.
4. Use the polynomial $(X^8 + X^9 + \dots + X^{30})^2 \cdot (1 + X + \dots + X^{30})$.

5. With a calculator, expand $(1 + \dots + X^{30})^3$, or consider all the triplets (a_1, a_2, a_3) where $a_i \geq 0$ and $a_1 + a_2 + a_3 = 30$. There are 31 possible choices for a_1 , then, once chosen a_1 , there are $31 - a_1$ choices for a_2 , and once fixed a_1 and a_2 , there is only 1 choice of a_3 . Hence, the total number of choices is $\sum_{a_1=0}^{30} (31 - a_1) \cdot 1 = \sum_{i=1}^{31} i = \frac{31 \cdot (31+1)}{2} = 31 \cdot 16 = 496$ total possible ways.

Exercise 4

[Radius of convergence and asymptotic behavior]

1. Let $r = \frac{1}{5}$. We have $a_n \leq \left(\frac{1}{r}\right)^n$ for all n , and r is the biggest number such that this holds. Besides, $A(z) = \sum_{n \geq 0} (5z)^n = \frac{1}{1-5z}$. For any $z > \frac{1}{5}$, we have $a_n z^n \not\rightarrow 0$ when $n \rightarrow +\infty$, so the series tends towards $+\infty$ (note that $A(z)$ is negative in this case, which would be a problem for a series counting positive integers); for any $z < \frac{1}{5}$, the series tends towards $A(z)$ (which is finite and positive). Note that $A(\frac{1}{5})$ is not well-defined.
2. By definition of r , for any z strictly bigger than r , we have that $a_n z^n \geq 1$ for infinitely many values of n . Consequently, the series $\sum_n a_n z^n$ is lower bounded by the sum of infinitely many 1s: it tends towards $+\infty$.
3. Let $z < r$, and pick n_o such that $a_n y^n \leq 1$ for any $n \geq n_o$:

$$\begin{aligned} \sum_n a_n z^n &= \sum_n a_n r^n \left(\frac{z}{r}\right)^n \\ &\leq \sum_{n=0}^{n_o} a_n z^n + \sum_{n \geq 0} 1 \cdot \left(\frac{z}{r}\right)^n \\ &= \sum_{n=0}^{n_o} a_n z^n + \frac{1}{1 - \frac{z}{r}} \end{aligned}$$

Hence the series $\sum_n a_n z^n$ converges (note that the left sum is finite). (To be really correct, we can define the radius to be the supremum of all y such that $a_n \leq \left(\frac{1}{y}\right)^n$ for n big enough, and re-do the above proof multiplying by $\frac{y^n}{y^n}$ for some y with $z < y < r$.)

4. Justifications are below the table: the main idea is to compute the minimum “pole” of $A(z)$, e.g. the minimum z such that $A(z) = +\infty$ or $A(z)$ contains the square root of a negative number, etc.

Sequence a_n	Justification	r^n
Exponential ρ^n for some $\rho > 0$	$A(\frac{1}{\rho}) = +\infty$	ρ^n
Fibonacci numbers	$\min \text{root} \geq 0 \text{ of } 1 - z - z^2 \text{ is } \frac{-1+\sqrt{5}}{2}$	$\left(\frac{2}{-1+\sqrt{5}}\right)^n = \left(\frac{1+\sqrt{5}}{2}\right)^n$
Catalan numbers	$\sqrt{1-4z}$ undefined for $z > \frac{1}{4}$	4^n
Motzkin numbers	$\sqrt{1-2z-3z^2}$ for $z > \frac{1}{3}$	3^n
Schröder numbers	$\sqrt{1-6z+z^2}$ undefined for $3-2\sqrt{2} < z < 3+2\sqrt{2}$	$\left(\frac{1}{3-2\sqrt{2}}\right)^n = (3+2\sqrt{2})^n$
Cayley numbers	see below (a)	e^n
Rooted trees with 2 or 3 children	see below (b)	$\leq 2.611^n$

(\cdot) Imagine that, for some function f , we have $f(A(z)) = z$, hence $A(z) = f^{-1}(z)$ **where the reciprocal f^{-1} is defined**. The smallest z such that f^{-1} is not-defined is the smallest z such that $f(z)$ is a value already attained by $f(x)$ for some $x < z$. This is the case for $z = z_o + \varepsilon$ where $f'(z_o) = 0$ and $\varepsilon > 0$ is arbitrarily small (make a drawing). Hence, the radius of $(a_n)_n$ is $r = \frac{1}{f'(z_o)}$ where z_o is the minimum value such that $f'(z_o) = 0$ (you actually need some more requirements, see “implicit functions” for the correct mathematical statements, but this is sufficient here).

(a) Here $f(x) = \frac{x}{e^x}$, and $f'(x) = \frac{1-x}{e^x}$, hence $f'(1) = 0$, and we have $f(1) = \frac{1}{e}$, so the radius is e .

(b) Here, $f(x) = \frac{x}{1+x^2+x^3}$, so $f'(x) = \frac{1-x^2-2x^3}{(1+x^2+x^3)^2}$. The polynomial $1-x^2-2x^3$ as only one real root z_o , approximately 0.657, and $\frac{1}{f'(z_o)} \approx 2.611$.

Exercise 5

[Bonus – Encoding a set through its sum]

For a set of numbers A , let $S(A) = \sum_{a \in A} a$ denote the sum of the numbers in A . For a set A of size at most 10, composed of numbers with 2 digits, we have: $S(A) \leq S(\{99, 98, 97, \dots, 90\}) = 945$. Now, let fix a set X of starting numbers, and consider all subsets $A \subseteq X$. As $|X| = 10$, there are $2^{10} = 1024$ such subsets A . If any subset A can be retrieved just by knowing its sum $S(A)$, that means that S is injective on the subsets of X (i.e. no two different subsets of X share the same sum). This implies that S maps $2^{10} = 1024$ elements (all the subsets of X) to at most 945 images (all the possible sums of a set of size at most 10). Hence, S cannot be injective, and it will be impossible to retrieve every subset A from the knowledge of X and $S(A)$, whatever the X chosen.

If $X = \{1, 2, 4, 8, 16, 32, 64\}$, not only $|X| = 7$, but also the starting numbers are the powers of two. As the writing of any number in binary form is unique and well-defined, the knowledge of the sum is enough to recover the subset (and all sums between 0 and 127 are possible).

You can now try to find X of sizes 8 and 9, containing 2-digit numbers, such that no two subsets share the same sum, or prove it does not exist.

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 10

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1

[Inversion]

Recall that $x^{\underline{n}} = x(x-1)(x-2)\dots(x-n+1)$.

For $k, n \in \mathbb{N}$, the *Stirling numbers of the first kind* $s(n, k)$ can be defined by the following polynomial equation

$$X^{\underline{n}} = \sum_{k=0}^n (-1)^{n-k} s(n, k) X^k.$$

1. Let $m, n \geq 1$. Show that the number of functions $f : [n] \rightarrow [m]$ is:

$$m^n = \sum_{k=0}^n S(n, k) m^{\underline{k}}.$$

2. Conclude from 1. that the Stirling numbers of the second kind satisfy the following polynomial equation:

$$X^n = \sum_{k=0}^n S(n, k) X^{\underline{k}}.$$

3. Prove *Stirling's inversion formula*: For two sequences $(u_n)_{n \geq 0}$, $(v_n)_{n \geq 0}$, we have:

$$v_n = \sum_{k=0}^n S(n, k) u_k \text{ for all } n \geq 0$$

if and only if

$$u_n = \sum_{k=0}^n (-1)^{n-k} s(n, k) v_k \text{ for all } n \geq 0.$$

4. Conclude that

$$\sum_{k \geq 0} S(n, k) (-1)^{k-m} s(k, m) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Exercise 2

[Words with requirements]

Consider the words (i.e. finite sequences) of length n we can write with the letters a, b, c , requiring that the number of a used is even, the number of b is odd, and the number of c can be whatever we want. Let w_n the number of such words, and $W(z) = \sum_{n \geq 0} w_n \frac{z^n}{n!}$ the associate exponential generating function.

1. Compute w_n for $n \leq 10$ (recall that 0 is even and not odd).
2. Let $A(z)$ be the exponential generating series of the number of words of length n with an even number of a (and no b nor c). Define $B(z)$ and $C(z)$ accordingly, and prove that $W(z) = A(z) \cdot B(z) \cdot C(z)$.
3. Show that $C(z) = e^z$, then $A(z) = \frac{1}{2}(e^z + e^{-z})$, and $B(z) = \frac{1}{2}(e^z - e^{-z})$.
4. Deduce $W(z)$ and w_n .

Exercise 3

[Cayley numbers]

Questions 1. and 2. are here for giving you a complete understanding of the subject: you will not earn points for doing them, and you will not loose points by not doing them. You can read through the text, and start at question 3.

Let \mathcal{C} be a collection of infinitely many distinct *objects* (e.g. graphs, sub-sets, paths) each of them having a certain *size* (e.g. number of vertices, cardinal, length) which is a integer number. Let \mathcal{C}_i be the collection of lists in \mathcal{C} of length i , and $c_i = |\mathcal{C}_i|$. Let $C(z) = \sum_{n \geq 0} c_n z^n$ be the associated (ordinary) generating function, and $\Gamma(z) = \sum_{n \geq 0} \frac{c_n}{n!} z^n$ the associated exponential generating series.

1. Show that the coefficient of z^n in the series $C(z)^k$ is the number of sequences of objects $(\alpha_1, \dots, \alpha_k)$ where $\alpha_i \in \mathcal{C}$, such that the sizes of $\alpha_1, \dots, \alpha_k$ sum to n .
2. Deduce that the coefficient of z^n in $\frac{1}{1-C(z)}$ counts the number of sequences of objects from \mathcal{C} whose sizes sum to n .
3. Show that the coefficient of $\frac{z^n}{n!}$ in the series $\frac{1}{k!} \Gamma(z)^k$ is the number of way to pick a partition of $[n]$ into k parts and to associate each part P of this partition to some $\alpha \in \mathcal{C}_{|P|}$. (Recall that parts in a partition are not ordered.)
4. Deduce that the coefficient on z^n in $e^{\Gamma(z)}$ is the number of ways to pick a partition of $[n]$ and associate each part P of this partition to some $\alpha \in \mathcal{C}_{|P|}$.

You have already seen applications of the above theorem on (ordinary) generating functions. Let a_n be the number of ways to construct a wall of length n and height 2 using dominoes, which can be placed vertically or horizontally. Here, a wall a sequence formed from the collection $\mathcal{C} = \{\square, \text{B}\}$. The size of a domino is its horizontal width (because we want a wall of width n): the size of B is 1, and \square is 2. Thus $C(z) = z + z^2$, and $A(z) = \frac{1}{1-C(z)} = \frac{1}{1-z-z^2}$ which is indeed the (ordinary) generating function of Fibonacci numbers.

The next questions are about an application of the second theorem.

5. Consider trees on vertices $[n]$ with one vertex marked in **red** (and the other in black). We call \mathcal{T} the collection of all labeled marked trees without size restriction. Show that there is a bijection between labeled marked trees on n vertices, and partitions of $[n-1]$ where each part P of the partition is associated with a tree on $|P|$ vertices.

6. Deduce that the exponential generating series $T(z)$ of labeled marked trees satisfies:

$$T(z) = ze^{T(z)}$$

One can deduce that there are n^{n-1} labeled marked trees on n vertices, and n^{n-2} labeled trees (where all the vertices have the same color).

Exercise 4

[Passwords with few zeros]

Let $f(n)$ be the number of numeric codes (i.e. passwords) of length n consisting only of the digits 0, 1, and 2, in which no two zeros occur consecutively.

1. Let $f(0) = 1$. Provide a recursive equation for $f(n)$, justify your answer.
2. Compute $\sum_{n \geq 0} f(n)x^n$. If you did not provide a recursive equation in part 1., alternatively use the equation $f(n) = 3f(n-1) + 2f(n-2)$ with $f(0) = 1$ and $f(1) = 2$. Note that this equation does not correspond to the solution from 1.!
3. Use your results from 2. to deduce a sequence a_n in terms of $f(n)$ satisfying

$$\sum_{n \geq 0} a_n x^n = \frac{1}{1 - 2x - 2x^2}$$

If you used the alternative equation in part 2., instead determine a sequence a_n in terms of $f(n)$ such that $\sum_{n \geq 0} a_n x^n = \frac{1}{1 - 3x - 2x^2}$.

Exercise 5

[Bonus – Fibonacci numbers and exam sheets]

You organize an exam with 20 questions. Each question can be answer either “True” or “False”. For each question, you grade the 17 712 students. At the end, you realize sadly that no students managed to answer correctly to 2 consecutive questions.

Something troubles you, however: two students gave exactly the same answers to all the questions... Is it normal, or should you look further into the details to see if they have cheated?

Hint: You should start by counting recursively the number of sequences in $\{0, 1\}^n$ with no consecutive 1s, then invoke an Italian mathematician born around 1170.

Submission: 02.07.2025 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

Exercise 7

[Even and odd parts]

We show in two ways that $[n]$ has as many sub-sets of even size, as it has sub-sets of odd size.

1. Show that $\sum_{A \subseteq [n], |A| \text{ even}} 1 - \sum_{A \subseteq [n], |A| \text{ odd}} 1 = \sum_{k=0}^n (-1)^k \binom{n}{k} = 0$, and conclude.
2. Fix $x \in [n]$. Show that the following application f is a bijection between the collection of even sub-sets and the collection of odd sub-sets:

$$f : A \mapsto \begin{cases} A \cup \{x\} & \text{if } x \notin A \\ A \setminus \{x\} & \text{if } x \in A \end{cases}$$

Exercise 8

[Number of solution of a degree 1 equation]

Let $a_{n,d}$ be the number of solutions of the equation $x_1 + \dots + x_n = d$, where x_i is a (non-negative) integer for each i . We want to compute $a_{n,d}$.

Let E be the set of solutions: $E = \{(x_1, \dots, x_n) \in \mathbb{N}^n ; x_1 + \dots + x_n = d\}$.

Let F be the set of strictly increasing sequences $0 < y_1 < \dots < y_n = n + d$.

Let G be the set of sub-sets of $\{1, \dots, n + d - 1\}$ of size $n - 1$.

1. Compute $a_{1,d}$, $a_{2,d}$, $a_{n,0}$ and $a_{n,1}$.
2. Find a bijection between E and F .
3. Find a bijection between F and G .
4. Compute $a_{n,d}$.

Discrete Mathematics – Solutions Sheet 10

Exercise 1

[Inversion]

1. The number of all functions $f : [n] \rightarrow [m]$ is given by m^n , as a result from the lecture. For the right-hand side of the given equality, we count the desired functions in another way. If $f : [n] \rightarrow [m]$ is a function, then f is surjective as a function from $[n]$ to $\text{Im}(f)$. The latter is always a non-empty subset of $[m]$. Classifying functions by their images, we get

$$\begin{aligned}
 m^n &= \sum_{A \subseteq [m]} |\{f : [n] \rightarrow A \text{ surjective}\}| \\
 &= \sum_{A \subseteq [m]} S(n, |A|) \cdot |A|! \\
 &= \sum_{k=0}^m \sum_{A \in \binom{[m]}{k}} S(n, k) \cdot k! \\
 &= \sum_{k=0}^m m(m-1) \cdots (m-k+1) S(n, k),
 \end{aligned}$$

which proves the assertion. Here, we used for the third equality that $S(n, 0) = 0$.

2. We note that both sides of the polynomial equation to be shown are polynomials in X of degree n . Two such polynomials are equal if they agree at $n+1$ points. From (i), the two polynomials already agree on \mathbb{N} , which proves the assertion.
3. $\{1, X, \dots, X^n\}$, as well as $\{1, X, X(X-1), \dots, X(X-1) \cdots (X-n+1)\}$, are each bases of the vector space of polynomials of degree $\leq n$. The corresponding basis transformations are achieved using the matrices $((-1)^{m-k} s(m, k))_{\substack{1 \leq m \leq n \\ 1 \leq k \leq n}}$ and $(S(m, k))_{\substack{1 \leq m \leq n \\ 1 \leq k \leq n}}$. The assertion thus follows directly from Theorem 2.9 (Inversion) of the lecture.
4. Since the matrices $T = ((-1)^{m-k} s(m, k))_{\substack{1 \leq m \leq n \\ 1 \leq k \leq n}}$ and $S = (S(m, k))_{\substack{1 \leq m \leq n \\ 1 \leq k \leq n}}$ are the two basis change matrices between the bases $\{1, X, \dots, X^n\}$ and $\{1, X, X(X-1), \dots, X(X-1) \cdots (X-n+1)\}$, they are inverses of each other and it holds that $S \cdot T = E_n$, where E_n denotes the $(n \times n)$ identity matrix. For the multiplication of the n -th row of S and the m -th column of T , we get directly

$$\sum_{k \geq 0} S(n, k) (-1)^{k-m} s(k, m) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Exercise 2

[Words with requirements]

- Here are all the possible words with ≤ 3 letters with these requirements:
 b ,
 bc, cb ,
 $aab, aba, baa, bcc, cbc, ccb, bbb$,
Hence, $w_0 = 0, w_1 = 1, w_2 = 2, w_3 = 7$.
- Consider only two requirements, say on a and on b , with associated exponential generating series $A(z)$ and $B(z)$. Then

$$A(z) \cdot B(z) = \sum_{p,m \geq 0} a_p b_m \frac{z^{p+m}}{p!m!} = \sum_{n \geq 0} \left(\sum_k a_k b_{n-k} \frac{n!}{k!(n-k)!} \right) \frac{z^n}{n!}$$

The middle parenthesis $\sum_k a_k b_{n-k} \binom{n}{k}$ counts the number of way to form a word of length n with both requirements on a and on b (first choose k , then choose which k among the n letters will be as , then complete by the number of words of length k with letters a , and the number of words of length $n - k$ with letters b). Consequently, $A(z) \cdot B(z) \cdot C(z) = W(z)$, as desired.

- We have $c_n = 1$ for all n , hence $C(z) = \sum_{n \geq 0} \frac{z^n}{n!} = e^z$. Besides, we have $a_n = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases} = \frac{1}{2}(1 + (-1)^n)$, hence we get the series $A(z) = \frac{1}{2} \sum_{n \geq 0} (1 + (-1)^n) \frac{z^n}{n!} = \frac{1}{2}(e^z + e^{-z})$. By the same method, we have $B(z) = \frac{1}{2}(e^z - e^{-z})$.
- Last but not least:

$$W(z) = e^z \cdot \frac{1}{2}(e^z + e^{-z}) \cdot \frac{1}{2}(e^z - e^{-z}) = \frac{1}{4}(e^{3z} - e^{-z})$$

Extracting the coefficient from $W(z)$, we get $w_n = \frac{1}{4}(3^n - (-1)^n)$. Please check that $n = 0, 1, 2, 3$ gives back the values of question 1.

Exercise 3

[Cayley numbers]

- Firstly, we have:

$$C(z)^k = \sum_{i_1, i_2, \dots, i_k \geq 0} c_{i_1} c_{i_2} \dots c_{i_k} z^{i_1 + i_2 + \dots + i_k} = \sum_{n \geq 0} \left(\sum_{i_1 + i_2 + \dots + i_k = n} c_{i_1} c_{i_2} \dots c_{i_k} \right) z^n$$

On the other hand, a sequence of length k of object in \mathcal{C} whose sizes sum to n is of the form $(\alpha_1, \alpha_2, \dots, \alpha_k)$, where $\alpha_i \in \mathcal{C}_i$ and $i_1 + i_2 + \dots + i_k = n$. Hence the number of such sequences is indeed counted by the big parenthesis above.

- We have $\frac{1}{1-C(z)} = \sum_{k \geq 0} C(z)^k$, so the coefficient on z^n in $\frac{1}{1-C(z)}$ counts the number of sequences of objects whose sizes sum to n , without any restriction on the length of the sequence.

3. Firstly, we have:

$$\begin{aligned}\Gamma(z)^k &= \sum_{i_1, i_2, \dots, i_k \geq 0} c_{i_1} c_{i_2} \dots c_{i_k} \frac{z^{i_1 + i_2 + \dots + i_k}}{i_1! i_2! \dots i_k!} \\ &= \sum_{n \geq 0} \left(\sum_{i_1 + i_2 + \dots + i_k = n} c_{i_1} c_{i_2} \dots c_{i_k} \frac{n!}{i_1! i_2! \dots i_k!} \right) \frac{z^n}{n!}\end{aligned}$$

On the other side, how many **ordered** partitions of $[n]$ with k parts associated to elements of \mathcal{C} are there? To construct such an ordered partition \mathcal{P} amounts to (1) the choice of a partition into parts P_1, P_2, \dots, P_k of cardinals i_1, i_2, \dots, i_k ; and (2) the choice of $\alpha_1 \in \mathcal{C}_{i_1}$, of $\alpha_2 \in \mathcal{C}_{i_2}$, ..., of $\alpha_k \in \mathcal{C}_{i_k}$.

The choice in (1) is counted by $\binom{n}{i_1} \binom{n-i_1}{i_2} \dots \binom{n-i_1-i_2-\dots-i_{k-1}}{i_k} = \frac{n!}{i_1! i_2! \dots i_k!}$.

The choice in (2) is counted by $c_{i_1} c_{i_2} \dots c_{i_k}$.

Hence, the coefficient on $\frac{z^n}{n!}$ in $\Gamma(z)^k$ counts the number of ordered partitions of $[n]$ with k parts together with an element of \mathcal{C} associated to each part so that the size of the element matches the cardinal of the part. Dividing by $k!$ amounts to un-ordering the partition.

4. We have $e^{\Gamma(z)} = \sum_{k \geq 0} \frac{1}{k!} \Gamma(z)^k$, so the coefficient on $\frac{z^n}{n!}$ in $e^{\Gamma(z)}$ counts the number of partitions of $[n]$ with objects associated with its parts (where size = cardinal), without restriction on the number of parts.
5. Consider a labeled marked tree t on n vertices, remove the red vertex, and, in the meanwhile, mark in red the neighbors of this red vertex. This yields a forest: equivalently a set of labeled marked trees. Looking at the vertex set of each tree in this forest, we get a partition of $[n-1]$ ("−1" because we removed the red vertex) where each part is associated to some labeled marked tree with as many vertices as elements in the part. Reciprocally, given a partition of $[n-1]$ and an association of each part P_i with a labeled marked tree t_i of size $|P_i|$, we can construct a labeled marked tree on n vertices by, first, adding a red vertex, then linking the marked vertex of each tree t_i to it. This gives the desired bijection (make drawings!).
6. According to question 2., the coefficient on z^{n-1} of $e^{T(z)}$ is the number of partitions of $[n-1]$ together with labeled marked trees associated to each parts. The coefficient on z^{n-1} of $\frac{1}{z} T(z)$ is the number of labeled marked trees on n vertices. According to question 5., these are the same numbers. Thus: $\frac{1}{z} T(z) = e^{T(z)}$.

Exercise 4

[Passwords with few zeros]

1. Let's consider the first digit. If it is 1 or 2, we can append it to any code of length $n-1$, giving $f(n-1)$ options. If it is 0, the next digit must be 1 or 2, and we can choose the rest in $f(n-2)$ ways. So:

$$f(n) = 2(f(n-1) + f(n-2)).$$

Also, $f(0) = 1$, $f(1) = 3$.

2. We have:

$$\begin{aligned}
\sum_{n \geq 0} f(n)x^n &= \sum_{n \geq 2} (2f(n-1) + 2f(n-2))x^n + f(0) + f(1)x \\
&= 2 \sum_{n \geq 2} f(n-1)x^n + 2 \sum_{n \geq 2} f(n-2)x^n + 1 + 3x \\
&= 2 \sum_{n \geq 1} f(n)x^{n+1} + 2 \sum_{n \geq 0} f(n)x^{n+2} + 1 + 3x \\
&= 2x \sum_{n \geq 1} f(n)x^n + 2x^2 \sum_{n \geq 0} f(n)x^n + 1 + 3x \\
&= 2x \sum_{n \geq 0} f(n)x^n - 2x + 2x^2 \sum_{n \geq 0} f(n)x^n + 1 + 3x \\
&= (2x + 2x^2) \sum_{n \geq 0} f(n)x^n + 1 + x.
\end{aligned}$$

Therefore:

$$(1 - 2x - 2x^2) \sum_{n \geq 0} f(n)x^n = 1 + x \quad \Leftrightarrow \quad \sum_{n \geq 0} f(n)x^n = \frac{1+x}{1-2x-2x^2}.$$

Alternative:

$$\begin{aligned}
\sum_{n \geq 0} f(n)x^n &= \sum_{n \geq 2} (3f(n-1) + 2f(n-2))x^n + f(0) + f(1)x \\
&= 3 \sum_{n \geq 2} f(n-1)x^n + 2 \sum_{n \geq 2} f(n-2)x^n + 1 + 2x \\
&= 3 \sum_{n \geq 1} f(n)x^{n+1} + 2 \sum_{n \geq 0} f(n)x^{n+2} + 1 + 2x \\
&= 3x \sum_{n \geq 1} f(n)x^n + 2x^2 \sum_{n \geq 0} f(n)x^n + 1 + 2x \\
&= 3x \sum_{n \geq 0} f(n)x^n - 3x + 2x^2 \sum_{n \geq 0} f(n)x^n + 1 + 2x \\
&= (3x + 2x^2) \sum_{n \geq 0} f(n)x^n + 1 - x.
\end{aligned}$$

Therefore:

$$(1 - 3x - 2x^2) \sum_{n \geq 0} f(n)x^n = 1 - x \quad \Leftrightarrow \quad \sum_{n \geq 0} f(n)x^n = \frac{1-x}{1-3x-2x^2}.$$

3. We have:

$$\begin{aligned}\frac{1}{1-2x-2x^2} &= \frac{1+x}{1-2x-2x^2} \cdot \frac{1}{1+x} \\ &= \sum_{n \geq 0} f(n)x^n \cdot \sum_{n \geq 0} (-1)^n x^n \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^n f(k)(-1)^{n-k} \right) x^n.\end{aligned}$$

So,

$$a_n = \sum_{k=0}^n f(k)(-1)^{n-k} = (-1)^n \sum_{k=0}^n f(k)(-1)^k.$$

Alternative:

$$\begin{aligned}\frac{1}{1-3x-2x^2} &= \frac{1-x}{1-3x-2x^2} \cdot \frac{1}{1-x} \\ &= \sum_{n \geq 0} f(n)x^n \cdot \sum_{n \geq 0} x^n \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^n f(k) \right) x^n.\end{aligned}$$

So,

$$a_n = \sum_{k=0}^n f(k).$$

Exercise 5

[Bonus – Fibonacci numbers and exam sheets]

We count recursively the number of sequences of 0 and 1 with no consecutive 1s. Let $S_n = \{\text{sequences of length } n \text{ with no consecutive 1s}\}$.

Consider such a sequence s of length n . We associate to s a sequence s' defined as: if s finishes by a 0, then s' is obtained from s by removing this last 0; if s finishes by a 1, then the digit before is a 0 (because there are no consecutive 1s), and s' is defined by removing 01 for the end of s . This association yields a bijection between S_n and the disjoint union $S_{n-1} \cup S_{n-2}$. Consequently, $|S_n| = |S_{n-1}| + |S_{n-2}|$. This is the Fibonacci recursion! As $|S_1| = 2$ and $|S_2| = 3$, we get that $|S_n| = F_{n+2}$ is the $(n+2)^{\text{th}}$ Fibonacci number.

In particular, $|S_{20}| = F_{22} = 17\,711$. As we have 17 712 exam sheets (which is quite a lot, I hope I have an automatic corrector), by the pigeon-hole principle, there are at least two exam sheets which yields the same sequence of True-False answers: two students who seem to have copied each other but did not.

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 11

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1

[Subsets with distances]

Let a_n be the number of subset $X \subseteq [n]$ such that for any $i, j \in X$, if $|j - i| < 3$, then $i = j$.

1. Find an equation involving some of the numbers $a_n, a_{n-1}, a_{n-2}, a_{n-3}$.
2. Deduce the (ordinary) generating function of the a_n .
3. By computing its derivative, show that the polynomial function $x \mapsto 1 - x - x^3$ has exactly 1 real root x_o .
4. Show: $0 < x_o < 1$. Deduce that the two other (complex conjugated) roots of $1 - x - x^3$ have a module > 1 . *Hint: It is an exercise on combinatorics...*

Exercise 2

[Catalytic variables and fountains]

A (n, k) -fountain is a collection of n circles of radius 1 (called *coins*), gathered into rows of tangent coins such that the bottom most row (called the *base*) is made of k tangent coins, and each other coin is tangent to exactly 2 coins of the previous row, see Figure 20.

Let $f_{n,k}$ be the number of (n, k) -fountains, let f_n be the number of fountains of n coins, and c_k be the number of fountains whose basis has k coins. We want to compute $F(x) = \sum_{n \geq 0} f_n x^n$.

1. Draw all fountains with 5 coins.
2. Let $g_{n,k}$ be the number of fountains with n coins and whose basis has k coins, and such that the second row contains $k - 1$ coins. Define the series $F(x, t) = \sum_{n,k \geq 0} f_{n,k} x^n t^k$ and $G(x, t) = \sum_{n,k \geq 0} g_{n,k} x^n t^k$. Show $g_{n,k} = f_{n-k, k-1}$, and deduce: $G(x, t) = xtF(x, xt)$.
3. By considering the first missing coin in the second row of a fountain, show that $F(x, t) = 1 + F(x, t) \cdot G(x, t)$.
4. Deduce that:

$$F(x, t) = \frac{1}{1 - \frac{xt}{1 - \frac{x^2 t}{1 - \frac{x^3 t}{\dots}}}} \quad \text{and} \quad F(x) = \frac{1}{1 - \frac{x}{1 - \frac{x^2}{1 - \frac{x^3}{\dots}}}}$$

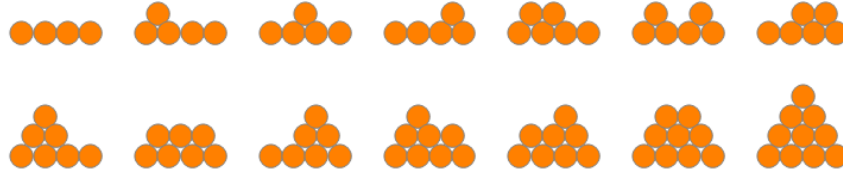


Abbildung 20: All fountains whose basis has 4 coins

5. Show that c_k is a Catalan number by constructing a bijection between fountains whose basis has k coins, and Dyck paths of a certain length. Deduce that $\frac{1}{1 - \frac{t}{1 - \frac{t}{1 - \frac{t}{\ddots}}}} = \frac{1 - \sqrt{1 - 4t}}{2t}$.

Hint: Consider the coins which are not covered by other coins, and link their centers.

Exercise 3

[Motzkin paths]

A *Motzkin path* of length n is a path $\mathcal{P} = ((0, y_0), (1, y_1), \dots, (n, y_n))$ from $(0, 0)$ to $(n, 0)$ where $y_i - y_{i-1} \in \{-1, 0, +1\}$, and $y_i \geq 0$ for all i . We denote by m_n the number of Motzkin paths of length n , and $M(z) = \sum_{n \geq 0} m_n z^n$ the associated generating series.

1. Draw all Motzkin paths of length 2, 3 and 4.
2. Show that there is a bijection between Motzkin paths of length n with $y_1 = 0$, and Motzkin paths of length $n - 1$.
3. Using the ideas you have seen on Dyck paths, show that there is a bijection between Motzkin paths with $y_1 > 0$, and pair of Motzkin paths $(\mathcal{Q}, \mathcal{Q}')$ where the length of \mathcal{Q} and \mathcal{Q}' sum to $n - 2$.

Hint: Consider $i_o \geq 1$ to be the smallest index such that $y_{i_o} = 0$, and split accordingly.

4. Deduce that $M(z) = z^2 M(z)^2 + z M(z) + 1$.
5. Solve the equation and obtain:

$$M(z) = \frac{1}{2z^2} \left(1 - z - \sqrt{1 - 2z - 3z^2} \right)$$

Exercise 4

[Non-self-intersecting paths]

Let a_n be the number of non-self-intersecting paths in \mathbb{Z}^2 with n steps, starting at the point $(0, 0)$, and allowing only steps in the directions North N , East E , and West W . ($N = (1, 0)$, $E = (0, 1)$, $W = (0, -1)$). Let $a_0 = 1$.

1. Show that $a_1 = 3$, and that for $n \geq 2$, we have:

$$a_n = 2a_{n-1} + a_{n-2}$$

2. Deduce polynomials $P(z)$, $Q(z)$, with $Q(z)$ of minimal degree, such that:

$$f_a(z) = \sum_{n \geq 0} a_n z^n = \frac{P(z)}{Q(z)},$$

3. Give an explicit formula for a_n .
4. How many paths are there if the restriction that paths must not intersect themselves is dropped?

Exercise 5

[Bonus – Decimal expansion]

For some random reason, you write $0.5 - \sqrt{0.249}$ in your favorite calculator, and you see appearing:

0.001 001 002 005 014 042 132 ...

As you have listen to the course very carefully, you realize this resemble the beginning of the sequence of Catalan numbers: 1, 1, 2, 5, 14, 42, 132, ...

Sadly, the 3 next digits of the decimal expansion of $0.5 - \sqrt{0.249}$ are 430, whereas the next Catalan number is 429.

Trying your best luck, you ask your calculator for $0.5 - \sqrt{0.2499}$, and see:

0.0001 0001 0002 0005 0014 0042 0132 0429 1430 4863 ...

All except the last 4 digits of the above decimal expansion match with Catalan numbers!

Explain why, in the decimal expansion of $0.5 - \sqrt{2499 \dots 99}$ (where there is n times the digit 9), you will see all Catalan numbers on $n + 2$ digits or less, except the last one.

Submission: 09.07.2025 before 23:59

Tutorium additional exercises

Exercise 6

[With ChatGPT]

Exercise 7

[Partitions and modulo]

Let $p(n)$ be the number of integer partitions of n (i.e. the number of ways to write n as a sum of integers). We want to prove that $p(5m+4)$ is divisible by 5 for all $m \geq 0$.

For two generating functions $A(z) = \sum_{n \geq 0} a_n z^n$ and $B(z) = \sum_{n \geq 0} b_n z^n$, we say that $A(z) \equiv B(z) [k]$ if, for all n , we have $a_n \equiv b_n [k]$ (where $a \equiv b [k]$ means that $b - a$ is divisible by k). The symbol \equiv between generating functions can be manipulated as usual: if $A(z) \equiv B(z) [k]$ and $C(z) \equiv D(z) [k]$, then $A(z) + C(z) \equiv B(z) + D(z) [k]$ and $A(z) \cdot C(z) \equiv B(z) \cdot D(z) [k]$, and so on. We will assume that all the infinite sums and products used in this exercise converges.

1. Recall the generating function of $p(n)$.
2. Write $\frac{1}{(1-z)^k}$ as a sum, then prove that $\frac{1}{(1-z)^5} \equiv \frac{1}{1-z} [5]$.
3. Deduce that $\frac{(1-z^5)(1-z^{10})(1-z^{15})\dots}{((1-z)(1-z^2)(1-z^3)\dots)^5} \equiv 1 [5]$.
4. Show that 5 divides the coefficient on z^{5m+5} in $z \cdot ((1-z)(1-z^2)\dots)^4$.
Very hard question
5. Using the next equation, prove that 5 divides $p(5m+4)$ for all $m \geq 0$:

$$\sum_{n \geq 1} p(n-1)z^n = z((1-z)(1-z^2)\dots)^4 \cdot \frac{(1-z^5)(1-z^{10})\dots}{((1-z)(1-z^2)\dots)^5} \cdot (1+z^5+z^{10}+\dots)(1+z^{10}+z^{20}+\dots)\dots$$

Exercise 8

[Dyck path area and pointed series]

The *area* of a Dyck path \mathcal{P} is the number of integer points in $\mathbb{N} \times \mathbb{N}$ that are under or on the Dyck path \mathcal{P} . For example, the path $((0,0), (1,1), (2,0))$ has area 4. We denote by b_n the sum over all Dyck paths of length $2n$, of their area.

1. Draw all Dyck paths of length $2 \cdot 3$, and compute their length.
2. Show that, equivalently, b_n is the number of couples (\mathcal{P}, x) where \mathcal{P} is a Dyck path, and x is a point of $\mathbb{N} \times \mathbb{N}$ under or on the path \mathcal{P} .
3. Write the usual recursive decomposition of Dyck paths, then write a version for the number of pairs (\mathcal{P}, x) where \mathcal{P} is a Dyck path and x a point under or on it.
4. Suppose a combinatorial class $\mathcal{C} = (\mathcal{C}_1, \dots)$ is counted by the generating series $C(z) = \sum_{k \geq 0} c_k z^k$. we define the *pointed class* \mathcal{C}^\bullet as $\mathcal{C}_k^\bullet = \{(\gamma, i) ; \gamma \in \mathcal{C}_k, i \in [k]\}$. Determine the generating series of \mathcal{C}^\bullet depending of $C(z)$.
(Hint: Derivatives are useful.)
5. Write a functional equation connecting the generating series $B(z)$ of the sequence $(b_n)_{n \in \mathbb{N}}$, and the generating series $D(z)$ of Dyck paths.
6. Give one (or more) proof(s) of the fact that $b_n = 4^n$, using a clever bijection.

7. (Re)deduce the series $D(z)$.

Exercise 9

[Narayana numbers]

For a Dyck path $\mathcal{P} = ((0, y_0), (1, y_1), \dots, (2n, y_{2n}))$, an index $i \in [1, 2n - 1]$ is a *peak* if $y_{i-1} < y_i > y_{i+1}$. We denote by $p(\mathcal{P})$ the number of peaks of the Dyck path \mathcal{P} . We want to determine the generating series in two variables $N(z, t) = \sum_{n \geq 0} \sum_{k \geq 0} N_{n,k} t^k z^n$, where $N_{n,k}$ is the number of Dyck paths of length $2n$ with k peaks. The numbers $N_{n,k}$ are called the Narayana numbers.

1. Compute the Narayana numbers $N(4, k)$ for all k , by drawing all involved Dyck paths.
2. For a Dyck path \mathcal{P} , let i_o be the smallest index such that $y_{2i_o} = 0$ and $i_o > 0$. Show that there is a bijection between Dyck paths \mathcal{P} of length $2n$ with k peaks and $i_o = n$, and Dyck paths of length $2(n - 1)$ with k peaks.
3. Show that there is a bijection between Dyck paths \mathcal{P} of length $2n$ with k peaks and $i_o < n$, and pairs of Dyck paths $(\mathcal{Q}, \mathcal{Q}')$ where the sum of the length of \mathcal{Q} and \mathcal{Q}' is $2(n - 1)$, and the sum of the number of peaks of \mathcal{Q} and \mathcal{Q}' is k , and \mathcal{Q}' is not of length 0.
4. Deduce that $N(z, t) = 1 + zN(z, t) + tzN(z, t)(N(z, t) - 1)$.
5. Solve the equation to obtain:

$$N(z, t) = \frac{1}{2tz} \left(1 + z(t - 1) - \sqrt{1 - 2z(t + 1) + z^2(t - 1)^2} \right)$$

6. Conclude that:

$$N_{n,k} = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$$

7. (Sanity check) When $t = 1$, one should obtain the usual Catalan numbers. Replace t by 1 in the equation above and verify that you get the equation for the Catalan generating series. Deduce $\sum_k \binom{n}{k} \binom{n}{k-1}$.

This last sum can be computed far more directly via Chu-Vandermonde identity.

Exercise 10

[Schröder paths]

Let f_n be the number of paths $((x_0, y_0), (x_1, y_1), \dots, (x_r, y_r))$ such that $(x_{i+1} - x_i, y_{i+1} - y_i) \in \{(1, 0), (1, 1), (0, 1)\}$, and $(x_0, y_0) = (0, 0)$, and $(x_r, y_r) = (n, n)$, and $x_i \geq y_i$ for all i . Such a path is called a Schröder path of size n . The number of them is denoted s_n , and the associated generating series is $S(z) = \sum_{n \geq 0} s_n z^n$.

1. Draw all Schröder paths of size $n \leq 4$.
2. Let i_o be the smallest index such that $i_o > 0$ and $x_{i_o} = y_{i_o}$. Show that there is a bijection between Schröder paths of size n with $i_o = 1$, and Schröder paths of size $n - 1$.
3. Show that there is a bijection between Schröder paths of size n with $i_o \neq 1$, and pairs of Schröder paths $(\mathcal{Q}, \mathcal{Q}')$ where the sum of their size is $n - 1$.
4. Deduce that $S(z) = zS(z)^1 + zS(z) + 1$.
5. Solve the equation to obtain:

$$S(z) = \frac{1}{2z^2} \left(1 - z - \sqrt{1 - 6z + z^2} \right)$$

Exercise 11

[Words avoiding words]

Fix an *alphabet* \mathcal{A} (i.e. a finite set) with $m = |\mathcal{A}| \geq 2$. A *word* on \mathcal{A} is an ordered sequence $\mathbf{p} = (p_1, p_2, \dots, p_r)$ where $p_i \in \mathcal{A}$. The *length* of \mathbf{p} is r .

1. How many words of length 5 are there on the alphabet containing 3 letters $\mathcal{A} = \{a, b, c\}$?
2. Let m be the cardinal of \mathcal{A} . How many words of length r are there.
3. Write the generating series $\sum_{r \geq 0} a_r z^r$ where a_r is the number of words of length r .

Fix a word \mathbf{p} of length r . We say that \mathbf{p} is i -self-correlated for some $0 \leq i < r$ if $(p_{i+1}, p_{i+2}, \dots, p_r) = (p_1, p_2, \dots, p_{r-i})$, where all the indices are considered modulo r . We denote $c_i = 1$ if \mathbf{p} is i -self-correlated, and $c_i = 0$ else; and construct the *polynomial of self-correlation* $c(z) = \sum_{i=0}^{r-1} c_i z^i$.

4. Compute the polynomial of self-correlation for the word $\mathbf{p} = (a, a, b, b, a, a)$.
5. Fix a word \mathbf{p} . Consider another word $\mathbf{q} = (q_1, \dots, q_s)$. The r -final sub-word of \mathbf{q} is (q_{s-r+1}, \dots, q_s) . Using $c(z)$, determine the generating series (per length) of the words \mathbf{q} on the alphabet \mathcal{A} such that \mathbf{p} is equal to the r -final sub-word of \mathbf{q} , but is not equal to any other sub-word of \mathbf{q} .
6. Show that the generating series (per length) of the words \mathbf{q} on the alphabet \mathcal{A} which avoid \mathbf{p} (i.e. \mathbf{p} is not a sub-word of \mathbf{q}) is:

$$S(z) = \frac{c(z)}{z^r + (1 - mz)c(z)}$$

Exercise 12

[Lagrange inversion theorem]

Let $F : z \mapsto \sum_{k \geq 0} f_k z^k$ be a function, with $f_0 = F(0) \neq 0$. Suppose the generating series $A(z) = \sum_{n \geq 0} a_n z^n$ satisfies $A(z) = F(A(z))$. We admit that, in this case, a_n is equal to the coefficient on z^{n-1} in $\frac{1}{n} F(z)^n$. This is called Lagrange inversion theorem.

1. Using Lagrange inversion, compute the number of labeled marked trees, whose exponential generating series $T(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n$ satisfies $T(z) = z e^{T(z)}$.
2. Recall that the generating series of Catalan numbers $C(z) = \sum_{n \geq 0} C_n z^n$ satisfies $C(z) = C(z)^2 + 1$. Using Lagrange inversion on the series $B(z) = C(z) - 1$, find (once again) the formula for Catalan numbers: $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Exercise 13

[Eulerian numbers]

Some more questions to come...

Let $A_{n,k}$ be the

Exercise 14

[Height on Dyck paths]

For a Dyck path $\mathcal{P} = ((0, y_0), (1, y_1), \dots, (2n, y_{2n}))$, the *height* of the path is defined as $h(\mathcal{P}) = \max_i y_i$. We want to determine the generating series in two variables $A(z, t) = \sum_{n \geq 0} \sum_{k \geq 0} a_{n,k} t^k z^n$, where $a_{n,k}$ be the number of Dyck paths of length $2n$ and height **at most** k .

1. For a Dyck path \mathcal{P} , let i_{\circ} be the smallest index such that $y_{2i_{\circ}} = 0$ and $i_{\circ} > 0$. Show that there is a bijection between Dyck paths \mathcal{P} of length $2n$ with height $\leq k$, and pairs of Dyck paths $(\mathcal{Q}, \mathcal{Q}')$ where the sum of the length of \mathcal{Q} and \mathcal{Q}' is $2(n-1)$, and $h(\mathcal{Q}) \leq h(\mathcal{P}) - 1$, and $h(\mathcal{Q}') \leq h(\mathcal{P})$.
2. Deduce that $a_{n,k} = \sum_{j=0}^{n-1} a_{j,k} a_{n-1-j,k-1}$.
3. *Some more questions to come...*

Discrete Mathematics – Solutions Sheet 11

Exercise 1

[Subsets with distances]

1. Let \mathcal{X}_n be the collection of all subsets $X \subseteq [n]$ such that for any $i, j \in X$, if $|i - j| < 3$, then $i = j$. Fix $X \in \mathcal{X}_n$: either $n \in X$ or $n \notin X$. If $n \in X$, then both $n - 1 \notin X$ and $n - 2 \notin X$. Let $Y = X \cap [n - 3]$, we have $X = \{n\} \cup Y$ and $Y \in \mathcal{X}_{n-3}$. Reciprocally, for all $Y \in \mathcal{X}_{n-3}$, we have $(\{n\} \cup Y) \in \mathcal{X}_n$. If $n \notin X$, then $X \in \mathcal{X}_{n-1}$. Hence there is a bijection between \mathcal{X}_n and $\mathcal{X}_{n-1} \cup \mathcal{X}_{n-3}$. This yields $a_n = a_{n-1} + a_{n-3}$, or equivalently $a_n - a_{n-1} - a_{n-3} = 0$.
2. Thanks to the theorem on linear recursions, we get that $A(z) = \sum_{n \geq 0} a_n z^n = \frac{P(z)}{1 - z - z^3}$, where P is a polynomial of degree 2 (or less). To get P , we need to compute $A(z)$ more in details (simply enumerating the possibilities gives $a_0 = 1, a_1 = 2, a_2 = 4$):

$$\begin{aligned}
 A(z) &= 1 + 2z + 4z^2 + \sum_{n \geq 3} (a_{n-1} + a_{n-3})z^n \\
 &= 1 + 2z + 4z^2 + z \sum_{n \geq 3} a_{n-1} z^{n-1} + z^3 \sum_{n \geq 3} a_{n-3} z^{n-3} \\
 &= 1 + 2z + 4z^2 + z(A(z) - 1 - 2z) + z^3 A(z) \\
 &= 1 + 2z + 4z^2 + z(A(z) - 1 - 2z) + z^3 A(z)
 \end{aligned}$$

$$\Rightarrow A(z) = \frac{1 + z + 2z^2}{1 - z - z^3}$$

3. The derivative of $x \mapsto 1 - x - x^3$ is $x \mapsto -1 - 3x^2$ which is negative for all $x \in \mathbb{R}$. Consequently, $x \mapsto 1 - x - x^3$ is strictly decreasing on \mathbb{R} . As this polynomial tends to $+\infty$ when $x \rightarrow -\infty$, and tends to $-\infty$ when $x \rightarrow +\infty$, it crosses the x -axis exactly once: the polynomial $1 - x - x^3$ has 1 real root.
4. For $x = 0$, we have $1 - x - x^3 = 1 > 0$, and for $x = 1$, we have $1 - x - x^3 = -1 < 0$, so, according to the previous question, the unique real root of $1 - x - x^3$ is between 0 and 1. Now, recall that $a_n = c_1 \alpha_1^n + c_2 \alpha_2^n + c_3 \alpha_3^n$ where α_1, α_2 and α_3 are the roots of the polynomial $1 - x - x^3$ (and c_1, c_2, c_3 are real numbers). If $|\alpha_1| < 1$ and $|\alpha_2| < 1$ and $|\alpha_3| < 1$, then $a_n \rightarrow 0$.

As we have established that the real root of $1 - x - x^3$ is of absolute value < 1 . Yet, $a_n \rightarrow +\infty$ (because it counts a number of sub-sets which clearly goes to infinity). Thus at least one of the complex roots is of module > 1 . As the complex roots are conjugated, they are both of module > 1 .

Exercise 2

[Catalytic variables and fountains]

1. There are 5 such fountains: 1 with 5 coins at its base, 3 with 4 coins at its base (and 1 coins in its second row), 1 with 3 coins at its base (and 2 coins in its second row).
2. Take a fountain with n coins whose basis has k coins and whose second row has $k - 1$ coins, then remove the basis: you get a fountain with $n - k$ coins whose basis has $k - 1$ coins (it is indeed a fountain because no coin was missing in the second row of the starting fountain). This defines a bijection between (n, k) -fountains whose second row is complete, and $(n - k, k - 1)$ -fountains. Hence $g_{n,k} = f_{n-k,k-1}$. Writing the generating series, we get (setting $j = k - 1$ and $m = n - k = n - j - 1$):

$$G(x, t) = \sum_{n,k} f_{n-k,k-1} x^n t^k = \sum_{m,j} f_{m,j} x^{m+j+1} t^{j+1} = xtF(x, xt)$$

3. Fix a (n, k) -fountain A . Consider the first missing coin of the second row of A , say the r^{th} one (for any $r \in [k]$): you can split A into a left part and a right part. The left part is formed by a basis of r coins together with all the coins which are above the r leftmost ones in the basis of A . The right part is formed by a basis of $n - r$ coins together with all the coins which are above the $n - r$ rightmost ones in the basis of A . As there is no r^{th} coin in the second row of A , these left and right part are disjoint (and their union is A). Consequently, this defines a bijection from non-empty (n, k) -fountains to pairs of fountains (B, B') such that B is a fountain whose second row is complete, and the sum of the number of coins of B and B' is n , and the sum of the size of the bases of B and B' is k . Thus, we get the equation on the level of generating functions (the $+1$ accounts for the empty fountain with 0 coins):

$$F(x, t) = F(x, t) \cdot G(x, t) + 1$$

4. Re-writing the last two equations give:

$$F(x, t) = \frac{1}{1 - x \cdot t \cdot F(x, xt)} = \frac{1}{1 - xt \cdot \frac{1}{1 - xt \cdot F(x, xt)}} = \dots$$

Continuing replacing $F(x, x^k t)$ iteratively, we get the desired expression. Besides, note that $F(x) = \sum_n f_n x^n = \sum_n (\sum_k f_{n,k} \cdot 1^k) x^n = F(x, 1)$. This gives the right expression.

5. For any (n, k) -fountains, linking the centers of the top coins draws a Dyck path (make a drawing) of length $2k$. This gives a bijection (don't forget to check the reciprocal) between fountains with basis k and Dyck paths. Hence the number of fountains with basis k is the k^{th} Catalan number. In particular, the given Figure depicts the case $k = 4$, and there are indeed $\text{Catalan}(4) = 14$ fountains with basis 4.

Besides, we know that $c_k = \sum_n f_{n,k}$, hence $F(1, t)$ is the generating series of c_k . As c_k are the Catalan number, we obtain $F(1, t) = \frac{1 - \sqrt{1 - 4t}}{2t}$. The left expression in 4. gives also $F(1, t) = \frac{1}{1 - \frac{t}{1 - \frac{t}{\dots}}}$.

Exercise 3

[Motzkin paths]

1. There are 2 paths of length 2, then 4 paths of length 3, and 9 paths of length 4 depicted in Figure 21.
2. If $y_1 = 0$, then the paths $((0, y_1), (1, y_2), \dots, (n-1, y_n))$ is again a Motzkin path (obtained by translating the starting Motzkin path to the left), because the we are still using the same steps.
 Reciprocally, taking a Motzkin path of length $n-1$, moving it in direction $(+1, 0)$, and appending a point $(0, 0)$ gives a Motzkin path of length n with y_1 .
 As these two constructions are inverse to each other, this describes the desired bijection.
3. Fix a Motzkin path \mathcal{P} of length n . Let i_o be the smallest index such that $y_{i_o} = 0$, and $\mathcal{Q}_o, \mathcal{Q}'$ the sub-paths of \mathcal{P} from 0 to i_o and from i_o to n respectively. As $y_1 > 0$, we know that $i_o > 1$, and in particular, \mathcal{Q}_o starts with a step $(+1, +1)$. Moreover, \mathcal{Q}_o finishes by a step $(+1, -1)$, by minimality of i_o . Hence, taking the sub-path \mathcal{Q} of \mathcal{Q}_o defined from 1 to $i_o - 1$ (and translating it by $(0, -1)$) we get a Motzkin path such that the sum of the lengths of \mathcal{Q} and \mathcal{Q}' is $n-2$.
 Reciprocally, taking two Motzkin paths whose lengths sum to $n-2$, we can construct a Motzkin path of length n by appending a step $(+1, +1)$ at the beginning of \mathcal{Q} , a step $(+1, -1)$ at its end, and concatenating the resulting path with \mathcal{Q}' .
 As these two constructions are inverse to each other, this describes the desired bijection.
4. A Motzkin path of length n is either the empty path, or a Motzkin path with $y_1 = 0$, or a Motzkin path with $y_1 > 1$. Thanks the two previous questions, we deduce that, up to bijection: a Motzkin path of length n is either the empty path, or a Motzkin path of length $n-1$, or a pair of Motzkin paths whose lengths sum to $n-2$. Hence the number m_n of Motzkin paths satisfies $m_0 = 1$, and $m_n = m_{n-1} + \sum_{i+j=n-2} m_i m_j$ for $n \geq 1$. Summing, we get: $M(z) = 1 + zM(z) + z^2M(z)^2$
5. Re-writing the above gives the polynomial equation of degree 2 (the variable is $M(z)$):

$$z^2M(z)^2 + (z-1)M(z) + 1 = 0$$

The discriminant is $\Delta(z) = (z-1)^2 - 4 \cdot z^2 \cdot 1 = -3z^2 - 2z + 1$. The solutions are $M(z) = \frac{-(z-1) \pm \sqrt{\Delta(z)}}{2z^2}$. For $z = 0$, we have $M(0) = m_0 = 1$. Hence the numerator shall tends to 0 when $z \rightarrow 0$ (for the fraction to converge). This implies that $M(z) = \frac{-(z-1) - \sqrt{\Delta(z)}}{2z^2}$, which is the desired formula.

Exercise 4

[Non-self-intersecting paths]

1. a_1 counts paths of exactly one step. There are 3 such paths (N, E, W). For the recurrence, we distinguish different types of paths of length n .

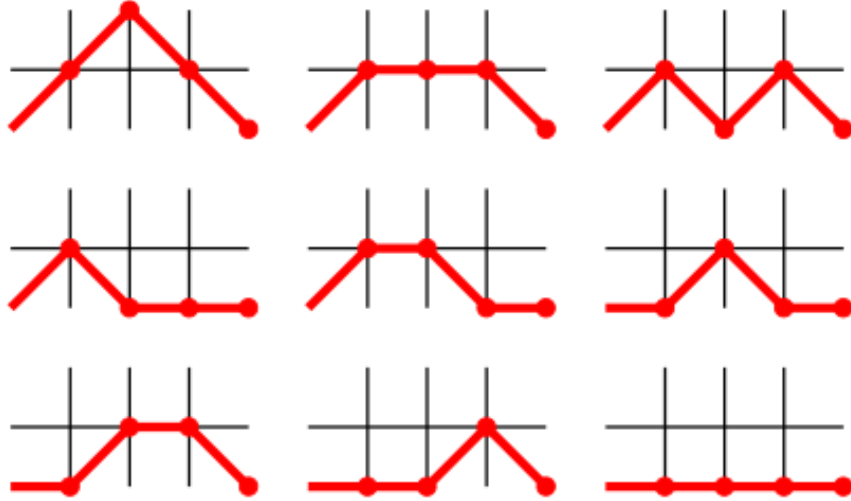


Abbildung 21: Motzkin paths of length 4

Type 1: Paths that start with N . In this case, the remaining path is an arbitrary path of length $n - 1$ satisfying the given conditions. There are a_{n-1} of these.

Type 2: Paths that start with EN , EE , or WW . These are in bijection with paths of length $n - 1$: one can shorten a path of length n to obtain a unique path of length $n - 1$, and vice versa. So there are also a_{n-1} such paths.

Type 3: Paths that start with WN . In this case, the remaining path is of length $n - 2$, satisfying the conditions. There are a_{n-2} of these.

Thus, the required recurrence follows.

2. The recurrence can be rewritten as

$$\begin{aligned} a_n &= 2a_{n-1} + a_{n-2} && \text{for } n \geq 2 \\ \Leftrightarrow a_{n+2} &= 2a_{n+1} + a_n && \text{for } n \geq 0 \\ \Leftrightarrow a_{n+2} - 2a_{n+1} - a_n &= 0 && \text{for } n \geq 0. \end{aligned}$$

By Theorem 11.18 (i), a_n has a rational generating function $f_a(z) = \sum_{n \geq 0} a_n z^n = \frac{P(z)}{Q(z)}$, with $Q(z) = 1 - 2z - z^2$ and $P(z)$ a polynomial of degree less than 2. We compute:

$$\begin{aligned} P(z) &= (a_0 + a_1 z + \text{higher terms})(1 - 2z - z^2) \\ &= a_0 + (-2a_0 + a_1)z \\ &= 1 + (-2 + 3)z = 1 + z. \end{aligned}$$

3. By Theorem 11.18 (iii), a_n has an explicit formula of the form

$$a_n = \sum_{i=1}^k P_i(n) \gamma_i^n.$$

First, determine the γ_i for $i = 1, \dots, k$. The roots of $Q(z)$ are found using the quadratic formula:

$$z_{1,2} = -1 \pm \sqrt{1^2 + 1} = -1 \pm \sqrt{2}.$$

The γ_i are the reciprocals of these roots:

$$\gamma_1 = \frac{1}{-1 + \sqrt{2}} = \frac{\sqrt{2} + 1}{1}, \quad \gamma_2 = \frac{1}{-1 - \sqrt{2}} = 1 - \sqrt{2}.$$

Alternative:

The reciprocal polynomial is

$$Q^R(z) = z^2 Q\left(\frac{1}{z}\right) = z^2 - 2z - 1,$$

with roots $z_{1,2} = 1 \pm \sqrt{2}$. These roots are precisely the γ_i , since:

$$Q(z) = (1 - (1 + \sqrt{2})z)(1 - (1 - \sqrt{2})z).$$

So, $\gamma_1 = 1 + \sqrt{2}$, $\gamma_2 = 1 - \sqrt{2}$, and

$$a_n = P_1(n)(1 + \sqrt{2})^n + P_2(n)(1 - \sqrt{2})^n,$$

where $P_1(n), P_2(n)$ are polynomials of degree ≤ 1 , i.e., constants $A, B \in \mathbb{C}$:

$$a_n = A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n.$$

Using initial values:

$$1 = a_0 = A + B$$

$$3 = a_1 = A(1 + \sqrt{2}) + B(1 - \sqrt{2})$$

Solving yields $B = 1 - A$, and:

$$3 = A(1 + \sqrt{2}) + (1 - A)(1 - \sqrt{2}) = 2\sqrt{2}A + 1 - \sqrt{2},$$

which gives:

$$A = \frac{2 + \sqrt{2}}{2\sqrt{2}} = \frac{1 + \sqrt{2}}{2}, \quad B = 1 - \frac{1 + \sqrt{2}}{2} = \frac{1 - \sqrt{2}}{2}.$$

So the explicit formula is:

$$a_n = \left(\frac{1 + \sqrt{2}}{2}\right) (1 + \sqrt{2})^n + \left(\frac{1 - \sqrt{2}}{2}\right) (1 - \sqrt{2})^n$$

for all $n \in \mathbb{N}$.

4. If the self-intersection restriction is removed, then each step has 3 possibilities, so the number of such paths is 3^n .

Exercise 5

[Bonus – Decimal expansion]

It sounds magic, but sadly it is not.

Let C_n be the n^{th} Catalan number. A number whose decimal expansion would group into blocks of 3-digits figuring the 7 first Catalan numbers is:

$$\frac{1}{1000}C_0 + \frac{1}{1000^2}C_1 + \frac{1}{1000^3}C_2 + \cdots + \frac{1}{1000^7}C_6$$

Hence, let's look at the series $\sum_{n \geq 0} \frac{1}{1000^{n+1}} C_n = \frac{1}{1000} \sum_{n \geq 0} C_n \left(\frac{1}{1000}\right)^n$. The later is the evaluation of $z \cdot C(z)$ at $z = \frac{1}{1000}$, where $C(z)$ is the generating series of Catalan numbers. From the course, you know that $C(z) = \frac{1 - \sqrt{1-4z}}{2z}$, so:

$$\sum_{n \geq 0} \frac{1}{1000^{n+1}} C_n = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{1000}}$$

Conversely, we have just proven that the decimal expansion of the real number $\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{1000}}$ (i.e. $0.5 - \sqrt{0.249}$) will display the Catalan numbers inside blocks of 3 digits, together with some messy digits after the $3 \cdot 7 = 21$ position after the coma. This displaying trick will work for any Catalan number which is on at most 3 digits, except if some bigger (4-digit) Catalan number generate an arithmetic carry: this is only the case for the biggest 3-digit Catalan number, i.e. 429 will not display correctly.

Of course, changing $\frac{1}{1000}$ by $\frac{1}{10^k}$ will display Catalan numbers in blocks of k digits, in the decimal expansion of $\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{10^k}} = 0.5 - \sqrt{0.2499 \dots 99}$ with $k - 2$ many 9s.

I invite you to trick your friends with other decimal expansions: it always sound very mystical at first glance!

Discrete Mathematics – Summer Semester 2025

Exercise Sheet 12

For each exercise, 10 points can be achieved. The Bonus Exercise can give up to +5 bonus points (so don't make it your priority). This is the English version of the same exercises as above, don't do them twice!

Exercise 1 [Exponential generating functions]

Remember that the exponential generating function associated to the sequence of numbers $(a_n)_{n \geq 0}$ is $A(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}$.

1. Show that $A'(z) = \sum_{n \geq 0} a_{n+1} \frac{z^n}{n!}$, where A' denote the usual derivative of A (with respect to z).
2. Give the exponential generating function of the number of permutations.
3. Let a_n be the number of permutations in S_n whose square is the identity, i.e. $\sigma(\sigma(i)) = i$ for all $i \in [n]$. Prove combinatorially: $a_{n+1} = a_n + na_{n-1}$.
4. Fix $\alpha, \lambda, \mu \in \mathbb{R}$, and derive $x \mapsto \alpha e^{\lambda x^2 + \mu x}$. Then find an explicit formula for the exponential generating function of $(a_n)_{n \geq 0}$ (you can consider that $a_0 = 1$).

Exercise 2 [Rencontres numbers]

A *fixed point* of a permutation $\sigma \in S_n$ is an index $j \in [n]$ satisfying $\sigma(j) = j$. Let $D_{n,k}$ be the number of permutations $\sigma \in S_n$ with k fixed points, called the *rencontres number*; and let $D(z, u) = \sum_{n \geq 0} \sum_{k \geq 0} D_{n,k} u^k \frac{z^n}{n!}$.

1. Show that $D_{n,k} = \binom{n}{k} D_{n-k,0}$.
2. Recall from the course (or re-prove): $\frac{1}{n!} D_{n,0} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}$.
Deduce that $\sum_{n \geq 0} D_{n,0} \frac{z^n}{n!} = \frac{e^{-z}}{1-z}$, and that $D(z, u) = \frac{e^{(u-1)z}}{1-z}$.
3. You've organized an exam for your students, but you are too lazy to correct, so you give back randomly the exam sheets to the students, and ask each student to correct the new sheet he/she has received. Assuming you have a lot of students, approximate the probability that no student received its own exam sheet.
4. Using a double counting argument, determine what is the average number of fixed point of the permutations on n elements.

Exercise 3 [Conjugation of permutations]

The *inverse* of a permutation $\sigma \in S_n$ is the permutation, denoted σ^{-1} , such that $\sigma(\sigma^{-1}(i)) = \sigma^{-1}(\sigma(i)) = i$ for all i .

1. For $\sigma, \tau \in S_n$, show that σ and $\tau\sigma\tau^{-1}$ have the same type.
2. Show that σ and σ^{-1} have the same type.
3. Deduce that for all $\sigma \in S_n$, there exists $\tau \in S_n$ such that $\sigma^{-1} = \tau\sigma\tau^{-1}$.

Exercise 4 [Bruhat graph and Steinhaus–Johnson–Trotter algorithm]
 The Bruhat graph Γ_n on permutations is defined as follow: the vertices are the permutations $\sigma \in S_n$, and there is an edge between σ and σ' if and only if there exists a transposition² $\tau = (i \ i+1)$ with $i \leq n-1$ such that $\sigma = \sigma'\tau$.

1. Draw Γ_2, Γ_3 .
2. How many vertices does Γ_n has? How many edges? Show that Γ_n is regular. Of which degree?
3. For which n is Γ_n Eulerian.
4. The Steinhaus–Johnson–Trotter algorithm works as follows: Start with the identity permutation $(1 \ 2 \ \dots \ n)$. Now find and transpose the largest possible number that can be transposed to its left or right, such that in each step, a new permutation is created that has not been encountered before. Show that Steinhaus–Johnson–Trotter algorithm enumerates **all** permutations of S_n , and deduce that Γ_n is Hamiltonian.

Exercise 5 [Bonus – Futurama’s theorem]

In the 10th episode of season 6 of Futurama, Pr. Farnsworth invents a machine that allows two minds to swap between two bodies. Thus, Amy’s mind ends up in Farnsworth’s body, and vice versa. The first problem is that if a swap is made, it is impossible to revert it (Farnsworth and Amy can no longer swap their minds back). The second problem is that very quickly, all the characters in the series use the machine: first Amy’s body is exchanged with Farnsworth’s body, then Bender’s body is exchanged with Amy’s body, Farnsworth’s body with Bender’s body, Leela with Farnsworth, Amy with the Robotic Bucket, Fry with Zelberg, Nikolaï with the Robotic Bucket, Hermes with Leela (Figure 22).

About ten minds are scattered across as many bodies... How to resolve this?

The solution is explained by the two mathematician-basketball players, Sweet Clyde and Bubblegum Tate: by adding only two people, it is possible to restore all minds to their original bodies, regardless of the number of swaps that have been made.

The theorem shows that if the bodies and minds of k people are mixed up, it takes only 2 more people and at most $k+3$ swaps to return everyone to their original state.

Prove this theorem, and show that, in the context of the series, the problem can be solved in 9 swaps without the intervention of any additional people.

It is indeed the proof of a new theorem in group theory. Completely written out. In a TV series.

Submission: 16.07.2025 before 23:59

²A transposition is a cycle of length 2.



Abbildung 22: The mind of the person on top (in the pink area) is now in the body of the person on the bottom. On this figure, the first 3 swaps have not been depicted.

Tutorium additional exercises

Exercise 6

[With ChatGPT]

Discrete Mathematics – Solutions Sheet 12

Exercise 1

[Exponential generating functions]

1. We have

$$\left(\sum_{n \geq 0} a_n \frac{z^n}{n!} \right)' = \sum_{n \geq 1} a_n \frac{z^{n-1}}{(n-1)!} = \sum_{n \geq 0} a_{n+1} \frac{z^n}{n!}$$

2. There are $n!$ permutations in S_n , hence the associated exponential generating function is $\sum_{n \geq 0} \frac{n!}{n!} z^n = \sum_{n \geq 0} z^n = \frac{1}{1-z}$.
3. Let $\sigma \in S_n$ be a permutation whose square is the identity. We distinguish between if $\sigma(n) = n$ or not. If $\sigma(n) = n$, then let $\sigma' \in S_{n-1}$ be defined as $\sigma'(i) = \sigma(i)$ for $i \in [n-1]$: the square of σ' is the identity if and only if the square of σ is too. If $\sigma(n) = k < n$, then consider σ'' to be the permutation of $[n] \setminus \{n, k\}$ defined by $\sigma''(i) = \sigma(i)$ for $i \notin \{n, k\}$: the square of the permutation σ'' is the identity if and only if the square of σ is too. Consequently, there is a bijection between $\{\sigma \in S_n ; \sigma^2 = \text{id}\}$ and $\{\tau \in S_{n-1} ; \tau^2 = \text{id}\} \cup \{(\gamma, k) \in S_{n-2} \times [n-1] ; \gamma^2 = \text{id}\}$. We deduce $a_n = a_{n-1} + (n-1)a_{n-2}$. A simple shift gives the desired formula.
4. Be careful, this is not a linear recurrence: some coefficient depends on n . Yet, passing the above formula to exponential generating functions, we get:

$$\begin{aligned} \sum_{n \geq 0} a_{n+1} \frac{z^n}{n!} &= \sum_{n \geq 0} a_n \frac{z^n}{n!} + \sum_{n \geq 0} a_{n-1} \frac{z^n}{(n-1)!} \\ A'(z) &= A(z) + z \sum_{n \geq 0} a_n \frac{z^n}{n!} \\ A'(z) &= A(z) + zA(z) \end{aligned}$$

Consequently, $A'(z) = (1+z)A(z)$. To solve this differential equation, either you know the theory of first order differential equations, or you can use the hint: let $f(x) = \alpha e^{\lambda x^2 + \mu x}$, then $f'(x) = \alpha(2\lambda x + \mu)e^{\lambda x^2 + \mu x} = (2\lambda x + \mu)f(x)$. In particular, for $\lambda = \frac{1}{2}$ and $\mu = 2$, and any α , we get the same equation as for A . Thus, $A(z) = \alpha e^{\frac{1}{2}z^2 + 2z}$ for some $\alpha \in \mathbb{R}$. Last but not least, to get the value of α , we evaluate: $A(0) = a_0 = 1$, and $A(0) = \alpha e^{\frac{1}{2} \cdot 0 + 0} = \alpha$.

Finally: $A(z) = e^{\frac{1}{2}z^2 + 2z}$.

Exercise 2

[Rencontres numbers]

1. There is a bijection between permutations $\sigma \in S_n$ with k fixed points and pairs (X, τ) where $X \in \binom{[n]}{k}$ and $\tau \in S_{n-k}$ with no fixed points: namely let X be the set of fixed points of σ , and τ be the permutation induced by σ on $[n] \setminus X$ (please, construct the reverse bijection). Consequently: $D_{n,k} = \binom{n}{k} D_{n-k,0}$.

2. The proof relies on the trivial fact that $\sum_k D_{n,k} = n!$, because $\sum_k D_{n,k}$ is the number of permutations with any number of fixed point. Recall that $\sum_{n \geq m} z^n = \frac{1}{1-z} z^m$. We have:

$$\sum_{n \geq 0} \frac{D_{n,0}}{n!} z^n = \sum_{n \geq k \geq 0} \frac{(-1)^k z^n}{k!} = \frac{1}{1-z} \sum_{k \geq 0} \frac{(-z)^k}{k!} = \frac{e^{-z}}{1-z}$$

Consequently (the last transformation is obtained via $p = n - k$):

$$\begin{aligned} e^{uz} \cdot \frac{e^{-z}}{1-z} &= \left(\sum_{k \geq 0} u^k \frac{z^k}{k!} \right) \cdot \left(\sum_{p \geq 0} D_{p,0} \frac{z^p}{p!} \right) \\ &= \sum_{p,k \geq 0} D_{p,0} \frac{1}{k!p!} u^k z^{k+p} \\ &= \sum_{n,k \geq 0} D_{n-k,0} \frac{n!}{k!(n-k)!} u^k \frac{z^n}{n!} \end{aligned}$$

Recalling $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we get $D(z, u) = \frac{e^{(u-1)z}}{1-z}$.

3. When you hand back the exam sheets to the students, you doing a permutations of the n exam sheets. A student received its own exam sheet if and only if it is a fixed point of such permutation. Hence the probability that no student receive its own sheet is the number of permutations with no fixed point, i.e. $D_{n,0}$, divided by the total number of permutations, $n!$. Consequently, we want the limit of $\frac{D_{n,0}}{n!} = \sum_{k=0}^n \frac{(-1)^k}{k!}$. Recall that $e^x = \sum_{k \geq 0} \frac{x^k}{k!}$, hence $\frac{D_{n,0}}{n!} \xrightarrow{n \rightarrow +\infty} e^{-1} \approx 0.368$. There is a bit more than 1 chance out of 3 that no student gets its own sheet.
4. Denote by $f(\sigma)$ the number of fixed points of a permutation $\sigma \in S_n$. We want to compute $m = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma)$. Let's double-count the set $\{(\sigma, x) ; \sigma \in S_n, x \in [n], \sigma(x) = x\}$. Firstly, $\sum_{\sigma \in S_n} \sum_{x \in [n], \sigma(x)=x} 1 = \sum_{\sigma \in S_n} f(\sigma) = m \cdot n!$. Secondly, for a fixed $x \in [n]$, the number of permutations $\sigma \in S_n$ such that $\sigma(x) = x$ is $(n-1)!$ because such a permutation is just a permutation of $[n] \setminus \{x\}$. Hence, $\sum_{x \in [n]} \sum_{\sigma \in S_n, \sigma(x)=x} 1 = \sum_{x \in S_n} (n-1)! = n!$. Finally: $m = 1$.

Exercise 3

[Conjugation of permutations]

Let $\sigma, \tau \in \mathbf{S}_n$ where τ is a transposition. We are interested in the cycle type of $s = \sigma\tau\sigma^{-1}$. Write $\tau = (ab)$. If $x \neq \sigma(a), \sigma(b)$, then $s(x) = x$ and $s(\sigma(a)) = \sigma(b)$ and $s(\sigma(b)) = \sigma(a)$, so $s = (\sigma(a)\sigma(b))$. In fact, for any permutation $\gamma \in \mathbf{S}_n$, we can write γ as a product of transpositions³ $\gamma = \tau_1\tau_2 \dots \tau_k$, and look at $s = \sigma\gamma\sigma^{-1}$:

$$s = \sigma\tau_1 \dots \tau_k\sigma^{-1} = (\sigma\tau_1\sigma^{-1})(\sigma\tau_2\sigma^{-1}) \dots (\sigma\tau_k\sigma^{-1})$$

³This also works with the cycle decomposition.

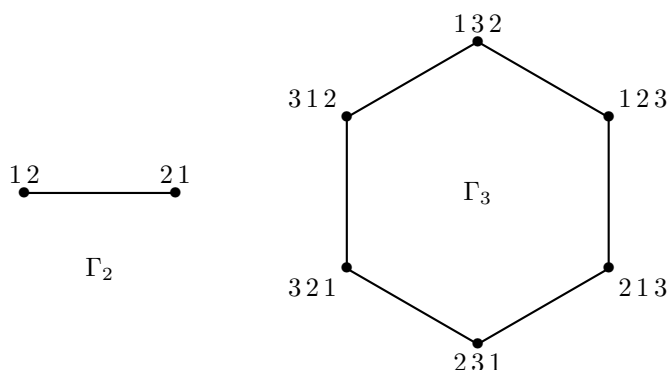
Indeed, by changing all $[1, n]$ that appear in the cycle decomposition of γ to $[\sigma(1), \sigma(n)]$, we obtain the cycle decomposition of $\sigma\gamma\sigma^{-1}$. Thus, the cycle type of γ has not been modified by the conjugation.

The inverse of a p -cycle is also a p -cycle (it is the same cycle written in reverse order), so by decomposing γ into a product of disjoint cycles, then inverting each of its cycles, we indeed obtain γ^{-1} with the same cycle type.

In fact, we showed something much more powerful earlier: we showed that two permutations that have the same cycle type are conjugate. Indeed, let two permutations of the same cycle type be γ, γ' , we write them as a product of cycles $\pi_1\pi_2\ldots\pi_k$ and $\pi'_1\pi'_2\ldots\pi'_k$, we define the permutation σ that maps π_1 to π'_1 , etc. (“associate” means that if $\pi_1 = (a_1 a_2 \ldots a_p)$ and $\pi'_1 = (b_1 b_2 \ldots b_p)$ then $\sigma(a_i) = b_i$). Once done, we have $\sigma\gamma\sigma^{-1} = \gamma'$.

In particular, σ^{-1} has the same cycle type as σ , so they are conjugated. This is quite a peculiar property and is not true in any group (for example, it is false in a cyclic group or a commutative group).

Exercise 4 [Bruhat graph and Steinhaus–Johnson–Trotter algorithm]



1. See figure.
2. Γ_n has $n!$ vertices. Each vertex has degree $n - 1$ because there are $n - 1$ transpositions of the form $(i \ i+1)$ with $i \leq n - 1$. Hence Γ_n is regular, and the degree formula gives us that Γ_n has $\frac{(n-1) \times n!}{2}$ edges.
3. For n odd, all the vertices of Γ_n have degree $n - 1$, which is even, hence Γ_n is Eulerian. For n even, the converse happens.
4. we are going to proceed by induction. For $n = 2$, Steinhaus–Johnson–Trotter algorithm starts at 12 , the biggest number that can be transposed (to its left or right) is 2 , and we get 21 , in which the biggest number that

can be transposed is again 2, and we are back at 12. We encourage the reader to run the algorithm for $n = 3$. Suppose the algorithm works for n (i.e. enumerates all permutations of \mathbf{S}_n when starting with $(1\ 2\ \dots\ n)$), we are going to show it works for $n + 1$. By definition, the algorithm only lists different permutations, so we have to show that it lists them all: we can either fix a permutation and prove that it is listed by the algorithm, or count the number of permutations listed in total and show it is $(n + 1)!$. We show the second. Take a step in the algorithm for $n + 1$, and suppose $n + 1$ can not be transposed. Then this step will just be a transposition of $i \leq n$ with $j \leq n$, that is to say it is a step the algorithm for n makes. Now, let's start at $1\ 2\ \dots\ n\ n + 1$ and run the algorithm. During the first n steps, the number $n + 1$ will be transposed to the left, until we get $n + 1\ 1\ 2\ \dots\ n$. At this point, $n + 1$ can not be transposed, so 1 step of the algorithm for n is performed. Then, $n + 1$ will be transposed to the right, until it reaches the rightmost position; then 1 step of the algorithm for n will be performed; then $n + 1$ will be transposed to the left until the leftmost position; and so on. Consequently, all the steps for of the algorithm for n will be performed, and n steps will be performed in between. In total, for each permutation that is yield by the algorithm for n , there are $n + 1$ permutations yield by the algorithm for $n + 1$. This gives a total of $(n + 1)!$ different permutations yield by the algorithm for $n + 1$: all permutations are enumerated. Besides, two permutations are consecutive in the enumeration given by the algorithm if and only if they differ by a transposition, so the enumeration of permutations given by algorithm for n is an Hamiltonian cycle (it comes back at the beginning after visiting all permutations) of the graph Γ_n .

Exercise 5

[Bonus – Futurama's theorem]

The solution is here :

<https://www.youtube.com/watch?v=J65GNFfL94c>